ON THE STABILITY OF SOME SYSTEMS OF THERMOELASTICITY OF TYPE III

BY

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Dedication

I solely dedicate this dissertation to Allah (Subhanahu wa ta'ala), the Lord of the Worlds. May He accept it from me as path to seeking knowledge. "Truly, my prayer and my service of sacrifice, my life and my death are (all) for Allah, the Cherisher of the Worlds".

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All praise is due to Allah (Subhanahu wa ta'ala), the absolute source of knowledge and wisdom, the One who by His blessing and favor, all good works are accomplished and perfected. I praise Him and seek His help and forgiveness. I bear witness that there is none worthy of being worshiped in truth except Him, and that prophet Muhammad (Salallaahu 'alayhi wa salaam) is His slave and messenger.

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THESIS ABSTRACT

NAME: Tijani Abdul-Aziz Apalara

TITLE OF STUDY: On the Stability of Some Systems of Thermoelasticity of Type III

MAJOR FIELD: Mathematics

DATE OF DEGREE: December 2013

In this dissertation, we study the well-posedness and the asymptotic behavior of some systems of thermoelasticity of type III in the presence of a viscoelastic damping and/or a delayed term. In this regard, we prove an exponential decay result for the one-dimensional case and several general decay results for the multi-dimensional case under some suitable assumptions. Furthermore, we investigate a Timoshenko-Thermoelastic system with second sound and delay and prove an exponential decay result under appropriate conditions on the delay and the structural parameters of the equations. In the absence of delay, we establish a polynomial decay result for the Timoshenko-Thermoelastic system with second sound. We use the multiplier method and/or convexity argument to establish the desired stability results of the systems.

الرسالة ملخص

الاسم: تيجاني عبد العيز أبلارا

عنوان الدراسة: حول استقرار بعض أنظمة المرونة الحرارية من النوع III

التخصص الاساسي: الرياضيات

تاريخ الشهادة: ديسمبر 2013

في هذه الرسالة، ندرس الصياغة الجيدة والسلوك المقارب لبعض أنظمة المرونة الحرارية من النوع الثالث في وجود حد لزوجة وحد تأخير. في هذا الصدد، نثبت نتيجة الاضمحلال الأسي في حالةالبعد الواحد والعديد من النتائج الاضمحلال العام في حالة تعدد الأبعاد تحت بعض الافتراضات المناسبة. وعلاوة على ذلك، نقوم بدراسة نظام تيموشينكومتزاوج مع مرونة حرارية بالصوت الثاني وفي ظل وجود حد تأخير وتثبت نتيجة الاضمحلال الأسي تحت شروط مناسبة على التأخير والمعاملات الهيكلية للمعادلات. وفي حالة غياب حد التأخير، نثبت نتيجة اضمحلال كثيرة حدود. لقد استخدمنا طريقة المضروبات وبعض خصائص التحدب للدوال لاثبات النتائج الاستقرارالمرجوة.

CHAPTER 1

INTRODUCTION

1.1 Thermoelasticity

In thermoelasticity theory, the deformation of a body is associated with a change of the heat content and, consequently, with a change of the body temperature. In other words, thermoelasticity deals with the study of the relationship between the elastic properties of a material and its temperature, or between its thermal conductivity and its stresses.

The theory of thermoelasticity goes back to the pioneer work of Duhamel [18], 1838, when he derived the equations relating the strain in an elastic body to the temperature gradient (the same results was obtained by Neumann [81] later in 1841). The theory was based on an independence assumption between the thermal effects and the mechanical effects. The total strain was determined by superimposing the elastic strain and the thermal expansion caused by the temperature distribution only.

In 1857, Thomson [114] used the laws of thermodynamics to determine the stresses and strains in an elastic body in response to varying temperatures. A century later, Landau and Lifshitz [50] in 1953 used classical thermodynamics methods to derive the coupled equations of thermoelasticity[1]. The derived equations coupled a hyperbolic equation and a parabolic equation in the same domain. In one space dimension (1−D), the basic linear equations are given by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \beta \theta_x = 0, \\ c_0 \theta_t - \kappa \theta_{xx} + \beta u_{tx} = 0 \end{cases} \tag{1.1}$$

and in $n-$dimensional space $(n \geq 2)$

$$\begin{cases} u_{tt} - \mu \Delta u - (\mu + \lambda) \nabla (\operatorname{div} u) + \beta \nabla \theta = 0, \\ c_0 \theta_t - \kappa \Delta \theta + \beta \operatorname{div} u_t = 0, \end{cases} \tag{1.2}$$

where $u = u(t,x) \in \mathbb{R}^m, m \geq 1$, and $\theta = \theta(t,x) \in \mathbb{R}$ denote the displacement and the temperature difference to the equilibrium state, respectively, t denotes the time-variable and x is the space-variable. Physical properties of the underlying isotropic medium are described by the thermal conductivity $\kappa > 0$, the elasticity modulus $\alpha = \lambda + 2\mu > 0$, or Lamé moduli μ and λ with $\mu > 0$, and the thermoelastic coupling coefficient $\beta \neq 0$. In the classical model for heat propagation, the heat flux is governed by Fourier's law of heat conduction, which states that the

[1] *The detail of the aforementioned historical reviews can be found in* [48]

heat flux is proportional to the gradient of temperature. i.e.

$$q + \kappa \nabla \theta = 0, \tag{1.3}$$

where θ is the temperature (difference to a fixed constant reference temperature), q is the heat flux vector and κ is the coefficient of thermal conductivity. It is obvious that the combination of (1.3) with the energy equation for a rigid conductor

$$\gamma \theta_t = -\operatorname{div} q \tag{1.4}$$

leads to the parabolic diffusion equation

$$\theta_t = k \Delta \theta,$$

where $k = \kappa/\gamma$ is the thermal diffusivity. Consequently, because of the parabolic nature of the equation, the model using the classic Fourier's law leads to the physical paradox of infinite speed of heat propagation. In other words, any thermal disturbance at one point will be instantaneously transferred to the other parts of the body. However, experiments have shown that heat conduction in some dielectric crystals at low temperatures propagates with a finite speed (see [42]). Similarly, working with very short laser pulses in laser cleaning of computer chips is free of this paradox (see the references in [101]). To overcome this physical paradox but still keeping the essentials of a heat conduction process, many theories have subsequently emerged. One of which is the advent of the second sound effects observed experimentally in materials at a very low temperature. Second sound effects arise when heat is transported by a wave propagation process instead of the usual diffusion.

Thermoelasticity with second sound (transportation of heat by wave-like propagation), first, arose in the work of Tisza in 1938 and Landau in 1941, when they separately studied heat waves in liquid helium II, and gave a prediction for its speed. Their predictions were later confirmed experimentally by Maurer and Herlin in 1949. From a theoretical point of view, Cattaneo in 1948 proposed a theory that account for the existence of second sound. The proposed theory suggests replacing the classic Fourier's law (1.3), by a modified law of heat conduction called Cattaneo's law

$$\tau q_t + q + \kappa \nabla\theta = 0, \tag{1.5}$$

or by more general heat-flux equation of Jeffreys type[2],

$$\tau q_t + q + \kappa \nabla\theta + \tau\kappa_1 \nabla\theta_t = 0. \tag{1.6}$$

In the above equations ((1.5) and (1.6)), the positive parameter τ represents the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. The constant $\kappa_1 > 0$ denotes the effective thermal conductivity. If $\kappa_1 = 0$, then (1.6) reduces to (1.5); i.e., Cattaneo's equation. When $\tau = 0$, (1.6) reduces to (1.3); i.e., Fourier's law. The obtained heat equation is of hyperbolic type and hence, automatically, eliminates the paradox of infinite

[2] *See* [10] *and* [42] *for references and more details.*

speeds.

Replacing (1.3) by (1.5) leads to the so-called thermoelasticity systems with second sound. In the linear $1-$D case, the system is given by

$$\begin{cases} u_{tt} - \alpha u_{xx} + \beta\theta_x = 0, \\ c_0\theta_t + \gamma q_x + \beta u_{tx} = 0, \\ \tau q_t + q + \kappa\theta_x = 0 \end{cases} \tag{1.7}$$

and in the $n-$D $(n \geq 2)$ case by

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \\ c_0\theta_t + \gamma \operatorname{div} q + \beta \operatorname{div} u_t = 0, \\ \tau q_t + q + \kappa\nabla\theta = 0, \end{cases} \tag{1.8}$$

where $q = q(t, x)$ denotes the heat flux vector, $\alpha, \mu, \lambda, \beta, \gamma, \delta, \tau$ and κ are positive constants. If $\tau = 0$, then we recover the classical thermoelasticity systems.

Green and Naghdi [23], suggested another method which eliminated the paradox of infinite speeds. They used an analogy between the concepts and equations of the purely thermal and the purely mechanical theories and arrived at three types of constitutive equations for heat flow in a stationary rigid solid labeled as types I, II and III. These types of constitutive equations are such that when the respective theories are linearized, type I leads to the usual heat conduction by Fourier law

(1.3), type II leads to a telegraph equation[3]

$$\theta_{tt} + \frac{1}{\tau}\theta_t = c^2 \Delta\theta, \tag{1.9}$$

which is hyperbolic and transmits waves at a finite speed c, and type III leads to an equation of Jeffrey's type (1.6). Both types II and III theories for heat flow in a stationary rigid solid accommodate finite wave speed, but only type II involves no energy dissipation.

In line with the classification employed in [23] for flow of heat in a stationary rigid solid, Green and Naghdi [24] derived the so-called classical thermoelasticity (or thermoelasticity of type I). However, the procedure they used for the development of constitutive equations is based on a procedure proposed in [22], which employs an entropy balance law and requires the satisfaction of the reduced energy equation before imposing any further restriction arising from the Second Law of thermodynamics. Furthermore, they considered another thermoelastic theory, the thermal part of which arises from transmission of heat as waves and is analogous to that of the response of an elastic material in a mechanical theory. They referred to it as thermoelasticity without energy dissipation (or thermoelasticity of type II), since it involves no energy dissipation. The models in $1-$D and $n-$D ($n \geq 2$) are

[3] *Telegraph equation*(1.9) *can simply be derived from* (1.4) *and* (1.5)

$$
\begin{cases}
u_{tt} - \alpha u_{xx} + \beta\theta_x = 0, \\
\\
\theta_{tt} - \kappa\theta_{xx} + \beta u_{ttx} = 0
\end{cases}
\tag{1.10}
$$

and

$$
\begin{cases}
u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \\
\\
\theta_{tt} - \kappa\Delta\theta + \beta \operatorname{div} u_{tt} = 0,
\end{cases}
\tag{1.11}
$$

respectively. Equations (1.10)-(1.11) permit propagation of waves without damping.

Later in [25], Green and Naghdi derived the thermoelasticity models of type III for isotropic media, using the constitutive equations developed in [23]. The system in $1-$D is given by

$$
\begin{cases}
u_{tt} - \alpha u_{xx} + \beta\theta_x = 0, \\
\\
\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{txx} + \beta u_{ttx} = 0,
\end{cases}
\tag{1.12}
$$

and in $n-$D $(n \geq 2)$ by

$$
\begin{cases}
u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \\
\\
\theta_{tt} - \kappa\Delta\theta - \delta\Delta\theta_t + \beta \operatorname{div} u_{tt} = 0.
\end{cases}
\tag{1.13}
$$

If $\delta = 0$, then the thermoelasticity models of type III, (1.12) and (1.13), become thermoelasticity models of type II, (1.10) and (1.11), respectively. For further historical reviews on thermoelasticity, we refer the readers to [11, 12, 41] and the references therein.

1.2 Porous-Thermoelasticity

In 1972, Goodman and Cowin [21] proposed an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced in the pioneered work of Nunziato and Cowin [88] in 1979 when they developed a nonlinear theory of elastic materials with voids. This representation (i.e the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction) introduces an additional degree of kinematic freedom and was employed previously by Goodman and Cowin [21] to develop a theory for flowing granular materials.

In 1983, Cowin and Nunziato [14] established the linear theory of elastic materials with voids. Later, Ieşan [37]-[39], and Ieşan and Quintanilla [40] added the temperature as well as the microtemperature elements to the theory. The importance of such materials could not be over-emphasized as it has resulted in the huge number of papers published in different fields of human endeavors most importantly, in petroleum industry, material science, soil mechanics, foundation engineering, powder technology, biology and others. For more details on the theory of elastic solids with voids, the reader is referred to [13, 98] and the papers cited therein.

1.3 Delay Differential Equations

It is generally known that many systems in science and engineering can be described by models that include past effects. These systems, where the rate of change in a state is not only determined by the present states but also by the past states, are described by delay differential equations (DDEs). In other words, DDEs are differential equations in which the derivatives of some unknown functions at present time depend on the values of the functions at previous times. They are also known as systems with aftereffect, hereditary systems, equations with deviating argument or differential-difference equations. Mathematically, a simple delay differential equation for $x(t) \in \mathbb{R}^n$ takes the form

$$\frac{d}{dt}x(t) = f(t, x_t),$$

where $x_t = \{x(\tau) : \tau \leq t\}$ represents the trajectory of the solution in the past. The functional operator f takes a time input and a continuous function x_t and generates a real number $\frac{d}{dt}x(t)$ as its output.

Examples of such equation include:

(1) discrete/constant delay $\frac{d}{dt}x(t) = f(t, x(t-\tau))$,

(2) time-varying delay $\frac{d}{dt}x(t) = f(t, x(t-\tau(t)))$, and

(3) distributed delay $\frac{d}{dt}x(t) = f\left(t, \int_0^\tau \mu(s)x(t-s)ds\right)$,

where τ is the delay in time. The study of DDEs started after the First World War due to the development and use of automatic control systems. A time delay arises because a finite time is required to sense information and then react to it. Minorsky [73] (see also Minorsky [74]) studied the control of the motion of a ship with movable ballast and made a realistic mathematical model which contained a delay (representing the time for the readjustment of the ballast) and observed that the motion was oscillatory if the delay was too large. Furthermore, it has been demonstrated in the area of automatic control that a relatively small delay may lead to instability or significantly deteriorated performances for the corresponding closed-loop systems. Nevertheless, in order to reliably analyze and design feedback controls for such systems, it is imperative to take delay into consideration.

Time delay has been widely studied in fields as diverse as biology [58], population dynamics [49], neural networks [5], feedback controlled mechanical systems [36], lasers [94]. Richard [103] mentioned some other interesting and challenging areas where delay are strongly involved. According to the causes of delays, we may roughly classify them as physically inherent delays (physical or biological systems), technological delays, transmission delays, and information delays.

Although a delay may cause instability, yet it can also cause boundedness and stability. Delay effects can also be exploited to control nonlinear systems as stated by Pyragas [96]. For instance, it is well known that the scalar equation

$$\frac{dx}{dt} = x^2(t) \text{ with } x(0) = 1$$

has a solution tending to infinity in finite time, but if $\tau(t)$ is positive for all $t \geq 0$, then

$$\frac{dx}{dt} = x^2(t - \tau(t))$$

has all solutions continuable for all $t \geq 0$ (See [7] page $245 - 246$ for details). In addition, it has been shown that voluntary introduction of delays can also benefit the control (see Richard [103] for many examples on this). Further exposition on delay equations can be found in [35, 87, 109].

1.4 Results Description

The aim of this dissertation is to investigate the well-posedness as well as the asymptotic behavior of solutions of some systems of thermoelasticity type III in the presence of viscoelastic damping and/or delay. In this regard, we study several problems and establish an exponential decay result for the one-dimensional case and several general decay results for the multi-dimensional case under some suitable assumptions. This study improves and generalizes several earlier results mentioned in Section 2.1, Section 2.2, Section 2.3 in the introduction, and in Section 4.1 and Section 7.1.

Our contributions start from Chapter three, where we investigate the asymptotic stability of solutions of the following thermoviscoelastic problem:

$$
\begin{cases}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\
+ \displaystyle\int_0^t g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \; x\in\Omega, \; t>0, \\
\theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta \operatorname{div} u_{tt}(x,t), = 0, \qquad x\in\Omega, \; t>0, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x\in\Omega, \\
\theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \qquad x\in\Omega, \\
u_t(x,-t) = f_0(x,t), \qquad x\in\Omega, \; t\in(0,\tau), \\
u(x,t) = \theta(x,t) = 0, \qquad x\in\partial\Omega, \; t\geq 0,
\end{cases}
\tag{1.14}
$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a boundary $\partial\Omega$ of class $C^2, u = u(x,t) \in \mathbb{R}^n, \; \theta(x,t) \in \mathbb{R}$, the relaxation function g is positive and decreasing, the coefficients $\mu, \lambda, \beta, \mu_1, \kappa, \delta$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. This is a thermoviscoelastic system of type III with a constant internal delay. The system is supplemented by initial data $u_0, u_1, \theta_0, \theta_1$ and a history function f_0. We consider (1.14) and establish a general decay result for the associated energy functional. This result extends the result obtained by Kirane and Said-Houari [44] for viscoelastic wave equation with a delay to a thermoelastic system of type III with delay. Furthemore, in contrast to [44], we do not require that μ_2 be positive.

In Chapter four, we study the well-posedness and the asymptotic behavior of the following thermoelastic system of type III with delay term and infinite memory:

$$
\begin{cases}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\
\quad + \displaystyle\int_0^\infty g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \text{ in } \Omega\times(0,\infty), \\
\theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta \operatorname{div} u_{tt}(x,t), = 0 \qquad \text{in } \Omega\times(0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x) \quad \text{in } \Omega, \\
u(x,-t) = f_0(x,t) \qquad \text{in } \Omega\times[0,\infty), \\
u_t(x,-t) = f_1(x,t) \qquad \text{in } \Omega\times(0,\tau), \\
u(x,t) = \theta(x,t) = 0 \qquad \text{on } \partial\Omega\times[0,\infty).
\end{cases}
\tag{1.15}
$$

We prove the well-posedness of (1.15) using semi-group theory, and a general decay result using the multiplier method and some convexity arguments similar to those in [28].

In Chapter five, we study the well-posedness as well as the asymptotic stability of the following one-dimensional system of thermoelasticity type III with delay:

$$
\begin{cases}
u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta\theta_x(x,t) + \mu u_t(x,t-\tau) = 0, & x \in (0,1),\ t > 0, \\
\theta_{tt}(x,t) - \kappa\theta_{xx}(x,t) - \delta\theta_{xxt}(x,t) + \beta u_{xtt}(x,t), = 0, & x \in (0,1),\ t > 0, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), x \in (0,1), & \\
u_t(x,-t) = f_0(x,t), & x \in (0,1),\ t \in (0,\tau), \\
u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, & t \geq 0.
\end{cases}
\tag{1.16}
$$

We establish the well-posedness using the semi-group method. We obtain an exponential decay result subject to the smallness of the weight μ of the delay. This stability result shows that the heat effect is strong enough to exponentially stabilize the system when the weight of the delay is small. Our result extends the result obtained by Quintanilla and Racke in [100] for system (1.16) without delay. We devoted Chapter six to the study of the asymptotic behavior of a one-dimensional porous-thermoelastic system of type III with a viscoelastic damping acting in one of the equations. Namely,

$$
\begin{cases}
\rho_1\varphi_{tt} - K(\varphi_x+\psi)_x + \theta_x = 0, & x\in(0,1),\ t>0,\\
\rho_2\psi_{tt} - \alpha\psi_{xx} + K(\varphi_x+\psi) - \theta + \displaystyle\int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & x\in(0,1),\ t>0,\\
\rho_3\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{xxt} + \beta\varphi_{xtt} + \beta\psi_{tt} = 0, & x\in(0,1),\ t>0,\\
\varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \psi(x,0)=\psi_0(x), \psi_t(x,0)=\psi_1(x), & x\in(0,1),\\
\theta(x,0)=\theta_0(x), \quad \theta_t(x,0)=\theta_1(x), & x\in(0,1),\\
\varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=\theta(0,t)=\theta(1,t)=0, & t\geq 0,
\end{cases}
\tag{1.17}
$$

where $\varphi(x,t)$ is the longitudinal displacement, $\psi(x,t)$ is the volume fraction, $\theta(x,t)$ is the difference temperature, the relaxation function g is positive and decreasing, the coefficients $\rho_1, \rho_2, \rho_3, K, \alpha, \kappa, \delta$ and β are positive constants, and $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0$ and θ_1 are initial data. We study system (1.17) and establish a general decay result for the case of equal as well as different speeds of wave propagation. This result extends the result obtained by Messaoudi and Fareh [70, 72] for classical porous thermoelastic system to a porous-thermoelastic system of type III.

Finally, in the last chapter (Chapter seven), we consider the following Timoshenko-thermoelastic system with second sound and delay:

$$
\begin{cases}
\rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu\varphi_t(x, t-\tau_0) = 0, x \in (0,1),\ t > 0, \\
\rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \qquad x \in (0,1),\ t > 0, \\
\rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \qquad x \in (0,1),\ t > 0, \\
\tau q_t + \beta q + \theta_x = 0, \qquad x \in (0,1),\ t > 0,
\end{cases}
\tag{1.18}
$$

together with the following initial, boundary and history conditions

$$
\begin{cases}
\varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x) \qquad x \in (0,1), \\
\psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ q(x,0) = q_0(x), \qquad x \in (0,1), \\
\varphi_t(x,-t) = f_0(x,t), \qquad x \in (0,1),\ t \in (0,\tau), \\
\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0,\ t \geq 0,
\end{cases}
\tag{1.19}
$$

where φ is the transverse displacement of the beam, ψ is the rotation angle of the beam, θ is the difference temperature, q is the heat flux, the coefficients $\rho_i,\ \beta, K,\ \delta, b, \tau$ are positive constants, μ is a real number, and $\tau_0 > 0$ represents the time delay. This is a thermoelastic system of Timoshenko type with a delay where the heat flux is given by Cattaneo's law. We study problem (1.18)-(1.19) and establish an exponential decay result under a smallness condition on the delay μ and a stability number introduced first by Santos *et al* in [108]. Furthermore, in the absence of a delay, we prove a polynomial decay result using multiplier method instead of the semigroup method used in [108]. Our result extends the result obtained by Santos *et al* in [108] for system (1.18) and (1.19) without delay.

1.5 Methodology

We use the multiplier method and/or convexity argument to establish the desired stability results of the systems. Multiplier method relies mostly on the construction of an appropriate Lyapunov functional $\mathcal{L}$ equivalent to the energy of the solution E. By equivalence $\mathcal{L} \sim E$, we mean

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0, \tag{1.20}$$

for two positive constants α_1 and α_2. To prove the exponential stability, we show that $\mathcal{L}$ satisfies

$$\mathcal{L}'(t) \leq -c_1 \mathcal{L}(t), \quad \forall t > 0, \tag{1.21}$$

for some $c_1 > 0$. A simple integration of (1.21) over $(0, t)$ together with (1.20) gives the desired exponential stability result.

In the case of general decay result, the obtained decay rate depends on the relaxation function g, which assume to satisfy the following two conditions:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \quad \mu - \int_0^\infty g(s)ds = l > 0,$$

where μ is a positive constant.

(A2) There exists a positive nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \le -\eta(t)g(t), \quad t \ge 0.$$

Then, we show that

$$(\eta(t)\mathcal{L}(t) + cE(t))' \le -k_0\eta(t)E(t), \ \forall t \ge t_0.$$

Thereafter, we exploit (1.20), to prove that

$$\mathcal{R}(t) = \eta(t)\mathcal{L}(t) + cE(t) \sim E(t). \tag{1.22}$$

Consequently, for some positive constant c_1, we obtain

$$\mathcal{R}'(t) \le -c_1\eta(t)\mathcal{R}(t), \ \forall t \ge t_0. \tag{1.23}$$

A simple integration of (1.23) over (t_0, t) together with (1.22) leads to the general decay result. In addition, we use a convexity argument, precisely in Chapter four, to obtain the desired decay result. In that chapter, g also must satisfy some other conditions (see page 64) in addition to (A1) stated above. For the well-posedness, we employ the standard semi-group theory.

1.6 Notation and some Useful Inequalities

Throughout this dissertation, we use the following standard $L^2(\Omega)$ and $H^1(\Omega)$ spaces equipped with their usual scalar products and norms denoted by

$$(u, v)_{L^2(\Omega)} = \int_\Omega u\, v\, dx, \qquad \|u\|^2_{L^2(\Omega)} = \int_\Omega |u|^2\, dx$$

and

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} = \int_\Omega uv\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx$$

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)},$$

respectively. The domain Ω is bounded in $\mathbb{R}^n (n \geq 2)$ with a smooth boundary. The space $H_0^1(\Omega)$ is defined to be

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \colon u = 0 \text{ on } \partial\Omega\},$$

where $\partial\Omega$ is the boundary of Ω. For $H^2(\Omega)$, we have

$$H^2(\Omega) = \{u \in H^1(\Omega);\ \frac{\partial u}{\partial x_i} \in H^1(\Omega) \quad \forall i = 1, 2, \cdots n\},$$

where

$$H^1(\Omega) = \Big\{ u \in L^2(\Omega); \exists g_1, g_2, \cdots g_n \in L^2(\Omega) : \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_0^\infty(\Omega) \Big\},$$

and $\dfrac{\partial u}{\partial x_i} = g_i$ in weak sense.

For the case of one dimension, we have

$$
\begin{aligned}
&(u,v)_{L^2(0,1)} = \int_0^1 u\,v\,dx, \qquad \|u\|^2_{L^2(0,1)} = \int_0^1 |u|^2\,dx,\\
&(u,v)_{H^1(0,1)} = (u,v)_{L^2(0,1)} + (u_x,v_x)_{L^2(0,1)} = \int_0^1 uv\,dx + \int_0^1 u_x v_x\,dx,\\
&\|u\|^2_{H^1(0,1)} = \|u\|^2_{L^2(0,1)} + \|u_x\|^2_{L^2(0,1)}.
\end{aligned}
$$

The space $H_0^1(0,1)$ is defined to be

$$H_0^1(0,1) = \{u \in H^1(0,1) \colon u(0) = u(1) = 0\},$$

where

$$H^1(0,1) = \left\{ u \in L^2(0,1); \exists g \in L^2(0,1) : \int_0^1 u\varphi' = -\int_0^1 g\varphi \quad \forall \varphi \in C_0^1(0,1) \right\},$$

and we called g the weak derivative of w (i.e. $u_x = g$).

Last, for $H^2(0,1)$ we have

$$H^2(0,1) = \{u, u_x, u_{xx} \in L^2(0,1)\}.$$

Any other spaces different from the above are stated in the chapters where they appeared. The following notations are used in the dissertation:

- $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + ... + \partial^2_{x_n}, \quad \operatorname{div} u = (\partial_{x_1} u_1 + \partial_{x_2} u_2 + ... + \partial_{x_n} u_n),$

- $\nabla = (\partial_{x_1},\ \partial_{x_2}, ..., \partial_{x_n})\,, \quad u_t = \dfrac{\partial u}{\partial t}, \quad u_x = \dfrac{\partial u}{\partial x}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2}, \quad u_{xx} = \dfrac{\partial^2 u}{\partial x^2},$

- $C^1(\Omega)$ denotes the space of all continuously differentiable functions on Ω,

- $C_0^1(\Omega)$ denotes the space of all continuously differentiable functions with compact support in Ω. The support of a continuous function f defined on Ω is the closure of the set of point where $f(x)$ is nonzero. That is

$$\text{supp}(f):\ = \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

Furthermore, we use c to denote a generic positive constant. The following inequalities are repeatedly used in the dissertation:

1. **Hölder's inequality**. Let $1 \leq p, q \leq \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

   By taking $p = q = 2$, we have the **Cauchy-Schwarz inequality**.

2. **Young's inequality**. Let $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \forall a, b \geq 0,$$

   where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. For $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

3. **Green's formula**. Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Then

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u\nabla v \cdot \nu ds, \qquad \forall u \in H^1 \text{ and } v \in H^2.$$

where ν is the outer unit normal to $\partial\Omega$. If $u \in H_0^1$, the Green's formula becomes

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx.$$

4. **Poincaré's inequality**. Let $1 \leq p < \infty$ and Ω be a bounded open subset of $\mathbb{R}^n$. Then there exists a constant C (depending on Ω and p only) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

If $p = 2$, then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Remark 1.1 *The Poincaré's inequality also holds for all $u \in W^{1,p}(\Omega)$ with*

$$\int_\Omega u dx = 0$$

provided that Ω is bounded.

Publications

The following results were Published/Submitted from our research:

(1) Tijani A. Apalara, Salim A. Messaoudi and Muhammad I. Mustafa, *Energy decay in thermoelasticity type III with viscoelastic damping and delay term*, EJDE, **2012**(2012), No. 128, pp. 115.

(2) Salim A. Messaoudi and Tijani A. Apalara, *Asymptotic stability of Thermoelasticity type III with delay term and infinite memory*, IMA J. Math. Cont. Info. (2013) doi:10.1093/imamci/dnt024

(3) Salim A. Messaoudi and Tijani A. Apalara, *General stability result in a memory-type porous thermoelasticity system of type III*, Arab J Math Sci (2013), http://dx.doi.org/10.1016/j.ajmsc.2013.08.004

(4) Salim A. Messaoudi and Tijani A. Apalara, *Well-posedness and exponential stability of a one-dimensional system of Thermoelasticity type III with delay* (submitted).

(5) Tijani A. Apalara and Salim A. Messaoudi, *An exponential stability result of a Timoshenko system with thermoelasticity with second sound and in the presence of delay* Applied Math. Optimization, (under 2nd Review).

CHAPTER 2

LITERATURE REVIEW

2.1 Time-Delay System

In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 113] and the references therein. The presence of delay may be a source of instability. See for example, Datko *et al*. [16], Nicaise and Pignotti [82], and Xu *et al*. [116], where it was proved that an arbitrarily small delay may destabilize a system, which is uniformly asymptotically stable in the absence of delay, unless additional conditions or control terms have been used. Datko *et al*. [16] examined the following problem:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, & x \in (0,1),\ t > 0, \\ u(0,t) = 0,\ u_x(1,t) = -ku_t(1,t-\tau), & t > 0, \end{cases}$$

where a, k, and τ are positive real numbers. By using spectral analysis method, they proved that, if k satisfies

$$0 < k < \frac{1 - e^{-2a}}{1 + e^{-2a}},$$

then the spectrum of this system lies in $Re\, w \leq -\beta$, where β is a positive constant depending on the delay τ and, consequently, the system is exponentially stable. As for the system of wave equation with a linear frictional damping term and a constant delay on the boundary;

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau), & x \in \Gamma_1,\ t > 0, \end{cases} \tag{2.1}$$

it is well known that in the absence of delay ($\mu_2 = 0, \mu_1 > 0$), system (2.1) is exponentially stable (see [47, 51, 52, 118]). In the presence of delay ($\mu_2 > 0$), Nicaise and Pignotti [82] proved, under the assumption $\mu_2 < \mu_1$, that the energy is exponentially stable. However, for the opposite case ($\mu_2 \geq \mu_1$), they were able to construct a sequence of delays for which the corresponding solution is unstable. Also, in [82], the authors obtained the same results for the case when both the damping and the delay act internally in the domain. That is,

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a(x)\left[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right] = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = 0, & x \in \Gamma_1,\ t > 0, \end{cases}$$

where $a \in L^\infty(\Omega)$ is a function satisfying specific conditions. See [2] for the treatment of this last system in more general abstract form. Nicaise and Pignotti [83] treated the situation when the constant delay in system (2.1) is replaced with a distributed delay of the form

$$\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x, t-s) ds$$

and established an exponential stability result under the condition that

$$\int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

The same result was also obtained by Nicaise *et al*. [86] when the term with constant delay on the boundary in (2.1) is replaced with the term

$$\mu_2 u_t(x, t-\tau(t)),$$

containing time-varying delay, under the assumption that

$$\mu_2 < \sqrt{1-d}\, \mu_1,$$

where d is a constant such that

$$\tau'(t) \leq d < 1, \ \forall t > 0.$$

Kirane and Said-Houari [44] considered a viscoelastic wave equation with delay of the form

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0,$$

for $x \in \Omega,\ t > 0$, together with initial and Dirichlet boundary conditions. They established general energy decay results under the condition that $\mu_2 \leq \mu_1$. In fact, the presence of a viscoelastic damping together with a frictional damping allowed the case $\mu_2 = \mu_1$ to be considered too. Said-Houari and Rahali [105] examined

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(x,t-s)ds + K(\varphi_x + \psi) + \mu_1 \psi_t(t) + \mu_2 \psi_t(t-\tau) = 0, \end{cases}$$

for $x \in (0,1), t > 0$, together with initial and Dirichlet boundary conditions and established the well-posedness as well as an exponential stability result of the energy for $\mu_2 \leq \mu_1$. Pignotti [95] considered the equation

$$u_{tt}(x,t) - \Delta u(x,t) + \mu_1 \chi_\omega u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad \text{in } \Omega \times (0,\infty),$$

where $\mu_1, \tau > 0$, and μ_2 is a real number, ω is the intersection between an open neighborhood of the set $\Gamma_0 = \{x \in \partial\Omega \colon m(x)\cdot\nu(x) > 0\}$ and Ω, $\nu(x)$ is the outer unit normal vector at a point $x \in \partial\Omega$, m is the standard multiplier and χ_ω is the characteristic function of ω. She established, under some geometric condition on the domain, the well-posedness of the problem and an exponential decay result for $|\mu_2| < \mu_1$. She thereby extended the previous results to the case when the weight of the delay is a real number.

Recently, Guesmia [31] considered the following second-order abstract linear equation with infinite memory as a dissipation and constant delay:

$$u_{tt} + Au - \int_0^\infty g(s)Au(t-s)ds + \mu u_t(t-\tau) = 0,\ \forall t > 0$$

and proved that the unique dissipation given by the memory term is strong enough to exponentially stabilize the system in the presence of a small delay. To the best of our knowledge, [31] is the first time an exponential decay result was obtained for a delay system without using any other source of damping (apart from the memory term) as in the case of the aforementioned papers.

In one dimensional space, Mustafa [79] studied a thermoelastic system with time-varying delay at the boundary, and showed that the damping effect through heat conduction is also strong enough to uniformly stabilize the system even in the presence of time-varying delay at the boundary. For more results concerning time delay, we refer the reader to [3, 17, 20, 80, 84, 85, 102] and the references therein.

2.2 Thermoelasticity of Type III

In this section, we recall some results regarding thermoelastic systems of type III. In one space dimension, Quintanilla and Racke [100] considered the equation

$$\begin{cases} u_{tt} - \alpha u_{xx} + \beta\theta_x = 0, \quad \text{in } (0,\infty)\times(0,L), \\ \theta_{tt} - \delta\theta_{xx} + \gamma u_{ttx} - \kappa\theta_{txx} = 0, \quad \text{in } (0,\infty)\times(0,L) \end{cases}$$

and used the spectral analysis method and the energy method to obtain the exponential stability for two types of boundary conditions (Dirichlet-Dirichlet or Dirichlet-Neuman). Furthermore, they proved an energy decay result for the radially symmetric situation in the multi-dimensional case ($n = 2, 3$).

Zhang and Zuazua [117] analyzed the long time behavior of the solution of the $n-$dimensional system

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \; x\in\Omega, \; t>0, \\ c\theta_{tt} - \kappa\Delta\theta - \delta\Delta\theta_t + \beta \operatorname{div} u_{tt} = 0, \qquad\qquad x\in\Omega, \; t>0, \end{cases}$$

together with initial and Dirichlet boundary conditions and showed that (i) for most domains, the energy of the system does not decay uniformly, (ii) under suitable conditions on the domain that may be described in terms of Geometric Optics, the energy of the system decays exponentially and (iii) for most domains in two space dimensions, the energy of smooth solutions decays polynomially.

Messaoudi and Soufyane [63] considered the system

$$
\begin{cases}
u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, & \text{in } \Omega\times\mathbb{R}_+,\\
\theta_{tt} - \kappa\Delta\theta - \delta\Delta\theta_t + \beta\operatorname{div} u_{tt} = 0, & \text{in } \Omega\times\mathbb{R}_+,
\end{cases}
$$

subject to a boundary feedback of viscoelastic type that acts on a part of the boundary and established exponential and polynomial stability results. This result was later generalized by Messaoudi and Al-Shehri [71] by taking a wider class of relaxation functions. They proved a more general decay result, from which the exponential and polynomial decay estimates are only special cases.

Recently, Qin and Ma [97] considered the system

$$
\begin{cases}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + \nabla\theta = 0, x\in\Omega,\ t>0,\\
\theta_{tt} - \Delta\theta_t - \Delta\theta + \operatorname{div} u_{tt} = 0, & x\in\Omega,\ t>0,\\
\theta = 0, & x\in\partial\Omega,\ t>0,\\
u = 0, & x\in\Gamma_0,\ t>0,\\
\dfrac{\partial u}{\partial\nu} - \int_0^t g(t-s)\Delta u(s)ds + H(u_t) = 0, x\in\Gamma_1,\ t>0
\end{cases}
$$

and established a general decay result depending on both g and H. This result extends the decay result obtained by Messaoudi and Mustafa [68] earlier for viscoelastic wave equations. We refer the reader to [56] for a global existence result for the higher-dimensional thermoviscoelastic equations and to [54, 62, 69] and the references therein for more results on Thermoelasticity type III.

2.3 Porous Thermoelasticity

In 1972, Goodman and Cowin [21] proposed an extension of the classical elasticity theory to porous media. The one-dimensional porous-elastic model has the form

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x, \\ \rho\kappa\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \tau\varphi_t - a\varphi, \end{cases} \tag{2.2}$$

where u is the longitudinal displacement, φ is the volume fraction of the solid elastic material, $\rho > 0$ is the mass density, $\kappa > 0$ is the equilibrated inertia and μ, α, τ, a are the constitutive constants which are positive and satisfy $\mu a > b^2$. This type of material has both macroscopic and microscopic structures. Scientists have investigated the coupling and its strength as well as the long-time behavior of solution, using dissipation mechanisms at the microscopic and/or macroscopic levels. The analysis of the decay rate for this class of materials was started by Quintanilla [99] when he considered (2.2) for $x \in (0, L),\ t > 0$, with initial and mixed boundary conditions, and showed that the damping in the porous equation $(-\tau\varphi_t)$ is not strong enough to obtain an exponential decay but only a slow decay can be obtained. To improve this decay, several other damping mechanisms were considered. Magańa and Quintanilla [60] investigated the temporal asymptotic behavior of the solutions of the one-dimensional porous-elasticity problem when several damping effects are present. They showed that viscoelasticity and temperature produce slow decay in time, and the same result was obtained when the porous-viscosity was combined with microtemperatures. However, when the viscoelasticity was coupled with porous damping or with microtemperatures, they proved that the decay was controlled by a negative exponential. We refer the reader to [76, 90] and the references therein for more results on asymptotic behavior in porous-elasticity with other forms of damping mechanisms.

For the case of the classical porous thermoelasticity, we mention the following model, which was introduced in [8],

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, & x \in (0, L),\ t > 0, \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + m\theta - \tau\varphi_t, & x \in (0, L),\ t > 0, \\ c\theta_t = k\theta_{xx} - \beta u_{xt} - m\varphi_t, & x \in (0, L),\ t > 0, \end{cases} \tag{2.3}$$

with initial and Dirichlet-Neumann boundary conditions. The function θ is the temperature difference, the coefficients $\rho, \mu, J, \alpha, \xi, \tau, c$ and k are positive constants. Casas and Quintanilla [8] considered the above system and used the semigroup theory and the method developed by Liu and Zheng [55] to establish the exponential decay of the solutions. Later, with $\tau = 0$ (absence of porous dissipation), the same authors showed in [9] that the heat effect alone is not strong enough to bring about an exponential decay but only a slow decay could be established. However, the heat effect together with microtemperature produced an exponential decay results. Similarly, when $\tau = 0$ and γu_{xxt} is added to the first equation in (2.3), Pamplona *et al.* [89] proved that the system lacks exponential stability but, by taking some regular initial data, a polynomial stability is obtained. Also, for $\tau = 0$, Soufyane *et al.* [112] considered (2.3) with the following boundary conditions:

$$\begin{cases} u(0,t) = \varphi(0,t) = \theta(0,t) = \theta(L,t) = 0, & t \geq 0, \\ u(L,t) = -\displaystyle\int_0^t g_1(t-s)\left[\mu u_x(L,s) + b\varphi(L,s)\right] ds, & t \geq 0, \\ \varphi(L,t) = -\alpha \displaystyle\int_0^t g_2(t-s)\varphi_x(L,s)ds, & t \geq 0, \end{cases}$$

where g_1 and g_2 are positive decreasing functions. They obtained a general decay result, from which the usual exponential and polynomial decay rates are just special cases. Soufyane [111] considered

$$\begin{cases} u_{tt} = u_{xx} + \varphi_x - \theta_x, & x \in (0,L),\ t > 0, \\ \varphi_{tt} = \varphi_{xx} - u_x - \varphi + \theta - \displaystyle\int_0^t g(t-s)\varphi_{xx}(s)ds, & x \in (0,L),\ t > 0, \\ \theta_t = \theta_{xx} - u_{xt} - \varphi_t, & x \in (0,L),\ t > 0, \end{cases} \tag{2.4}$$

with some initial and Dirichlet boundary conditions and g is a positive nonincreasing function. He used the multiplier technique to establish exponential and polynomial stability results depending on the relaxation function g. Recently, Messaoudi and Fareh [70, 72] considered

$$\begin{cases} \rho_1 u_{tt} - k(u_x + \varphi)_x + \theta_x = 0, & x \in (0,1),\ t > 0, \\ \rho_2 \varphi_{tt} - \alpha\varphi_{xx} + k\left(u_x + \varphi\right) - \theta + \displaystyle\int_0^t g(t-s)\varphi_{xx}(x,s)ds = 0, & x \in (0,1),\ t > 0, \\ \rho_3\theta_t - \kappa\theta_{xx} + u_{xt} + \varphi_t = 0, & x \in (0,1),\ t > 0, \end{cases} \tag{2.5}$$

with some initial and Dirichlet boundary conditions, where $\rho_1, \rho_2, \rho_3, k, \alpha, \kappa$ are positive constants and g is a positive decreasing function. They established some general decay results for the solutions in case of equal wave speeds $\left(\frac{\kappa}{\rho_1} = \frac{\alpha}{\rho_2}\right)$ as well as for different speeds of wave propagation $\left(\frac{\kappa}{\rho_1} \neq \frac{\alpha}{\rho_2}\right)$.

For nonclassical porous thermoelasticity, Magaña and Quintanilla [59] investigated the asymptotic behavior of the solutions of the following one-dimensional porous-thermo-elasticity problem:

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\left(\theta + \alpha\theta_t\right)_x, & x \in (0,\pi),\ t > 0, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi + m\left(\theta + \alpha\theta_t\right) - \tau\varphi_t, & x \in (0,\pi),\ t > 0, \\ h\theta_{tt} = k\theta_{xx} - \beta u_{xt} - m\varphi_t - d\theta_t, & x \in (0,\pi),\ t > 0 \end{cases} \tag{2.6}$$

with initial and mixed boundary conditions. They proved that, generally, the thermal damping ($\tau = 0$) is not sufficiently strong to guarantee the exponential decay of solutions. But when the porous dissipation ($\tau > 0$) is also present, the solutions decay exponentially. The arguments they used to prove the slow decay work only on a particular class of boundary conditions. However, for exponential decay of the solutions, the boundary conditions could be extended to other classes of boundary conditions. Leseduarte *et al*. [53] investigated the asymptotic behavior of solutions of thermo-porous-elasticity when the only dissipation mechanism present in the system is the porous dissipation. That is, they considered

$$
\begin{cases}
\rho u_{tt} = \mu u_{xx} + \gamma \varphi_x - \beta \psi_{xt}, & x \in (0, \pi),\ t > 0, \\
J \varphi_{tt} = b \varphi_{xx} + m \psi_{xx} - \xi \varphi + d \psi_t - \tau \varphi_t - \gamma u_x, & x \in (0, \pi),\ t > 0, \\
a \psi_{tt} = k \psi_{xx} + m \varphi_{xx} - d \varphi_t - \beta u_{xt}, & x \in (0, \pi),\ t > 0,
\end{cases} \tag{2.7}
$$

with initial and boundary conditions, where ψ is the thermal displacement. They showed that when the parameters m and β are not both zero, the decay of solutions is exponentially stable. Whereas, if one of the parameters m or β vanishes, the decay of solutions is slow in the sense that it cannot be controlled by a negative exponential (generally). We refer the reader to [13, 91] and the references therein for more results.

CHAPTER 3

THERMO-VISCO-ELASTICITY OF TYPE III WITH A DELAY TERM

In this chapter, we consider the following problem:

$$
\begin{cases}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\
\ + \displaystyle\int_0^t g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0,\ x\in\Omega,\ t>0, \\
\theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta\operatorname{div} u_{tt}(x,t), = 0, \qquad x\in\Omega,\ t>0, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x),\ x\in\Omega, \\
u_t(x,-t) = f_0(x,t), \qquad x\in\Omega,\ t\in(0,\tau), \\
u(x,t) = \theta(x,t) = 0, \qquad x\in\partial\Omega,\ t\geq 0,
\end{cases}
\tag{3.1}
$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a boundary $\partial\Omega$ of class $C^2, u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta(x,t)$ is the difference temperature, the relaxation function g is positive and decreasing, the coefficients $\mu, \lambda, \beta, \mu_1, \kappa, \delta$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. This is a (type III) thermoelastic system with the presence of a viscoelastic damping and a constant internal delay supplemented by initial data $u_0, u_1, \theta_0, \theta_1$ and a history function f_0. In section 3.1, we introduce some transformations and assumptions needed in the chapter. Some technical lemmas and the statement with proof of the main results are given in section 3.2 and section 3.3, respectively. Finally, we give some examples to illustrate our results.

3.1 Assumptions and Transformations

In this section, we present some material needed in the proof of our results. For the relaxation function g, we assume the following:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \quad \mu - \int_0^\infty g(s)ds = l > 0.$$

(A2) There exists a positive nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\eta(t)g(t), \quad t \geq 0.$$

Remark 3.1 There are many functions that satisfy (A1) and (A2). Below are three examples of such functions with the assumptions that $a, b > 0$ and $a < \mu b$.

1. If $g(t) = ae^{-bt}$, then $g'(t) = -\eta(t)g(t)$, where $\eta(t) = b$.

2. If $g(t) = \frac{a}{(1+t)^{b+1}}$, then $g'(t) = -\eta(t)g(t)$, where $\eta(t) = \frac{b+1}{1+t}$.

3. If $g(t) = \frac{a}{(e+t)[ln(e+t)]^{b+1}}$, then $g'(t) = -\eta(t)g(t)$, where

$$\eta(t) = \frac{1}{e+t} + \frac{b+1}{(e+t)\ln(e+t)}.$$

In order to exhibit the dissipative nature of system (3.1), it is convenient to introduce, as in [117], the new variable

$$v(x,t) = \int_0^t \theta(x,s)ds + \chi(x), \tag{3.2}$$

where $\chi(x)$ is the solution of

$$\begin{cases} -\kappa\Delta\chi = \delta\Delta\theta_0 - \theta_1 - \beta \operatorname{div} u_1, & \text{in } \Omega, \\ \chi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Integrating the second equation in (3.1) with respect to t and using the fact that $v_t = \theta,\ \ v_{tt} = \theta_t$, we have

$$v_{tt} - \kappa\Delta v - \delta\Delta v_t + \beta \operatorname{div} u_t = \theta_1 - \delta\Delta\theta_0 + \beta \operatorname{div} u_1 - \kappa\Delta\chi(x),$$

then using (3.3), we get

$$v_{tt} - \kappa \Delta v - \delta \Delta v_t + \beta \operatorname{div} u_t = 0.$$

Next, as in [82], we introduce another new dependent variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

A simple differentiation shows that z satisfies

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \qquad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Hence, problem (3.1) is equivalent to the following:

$$
\begin{cases}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla v_t(x,t) \\
\quad + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0, & x\in\Omega,\ t>0, \\
v_{tt}(x,t) - \kappa\Delta v(x,t) - \delta\Delta v_t(x,t) + \beta \operatorname{div} u_t(x,t) = 0, & x\in\Omega,\ t>0, \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x\in\Omega, \rho\in(0,1),\ t>0, \\
z(x,0,t) = u_t(x,t), & x\in\Omega,\ t>0, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x\in\Omega, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x\in\Omega, \\
z(x,\rho,0) = f_0(x,\tau\rho), & x\in\Omega,\ \rho\in(0,1), \\
u(x,t) = v(x,t) = 0, & x\in\partial\Omega,\ t\geq 0.
\end{cases}
\tag{3.4}
$$

The dissipative nature of (3.4) is explicitly seen at the level of the energy E. In fact, we easily get $E' \leq 0$ (see (3.7)). Thus, we will consider problem (3.4) instead of (3.1). In what follows, we consider (u,v,z) to be a solution of system (3.4) with the regularity needed to justify the calculations in this chapter. The existence and uniqueness of strong and weak solutions of system (3.4) can be proved by repeating the arguments of [44].

Now, we assume that $|\mu_2| \leq \mu_1$ and let ξ be a positive constant satisfying

$$
\begin{cases}
\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), & \text{if } |\mu_2| < \mu_1, \\
\xi = \tau\mu_1, & \text{if } \mu_1 = |\mu_2|.
\end{cases}
\tag{3.5}
$$

The energy associated with problem (3.4) is

$$E(t)=\frac{1}{2}\int_{\Omega}|u_t|^2dx+\frac{1}{2}\int_{\Omega}v_t^2dx+\frac{1}{2}\Big(\mu-\int_0^t g(s)ds\Big)\int_{\Omega}|\nabla u|^2dx+\frac{\kappa}{2}\int_{\Omega}|\nabla v|^2dx$$
$$+\frac{(\mu+\lambda)}{2}\int_{\Omega}(\operatorname{div}u)^2dx+\frac{1}{2}(g\circ\nabla u)(t)+\frac{\xi}{2}\int_{\Omega}\int_0^1 z^2(x,\rho,t)\,d\rho\,dx, \tag{3.6}$$

where

$$(g\circ\nabla u)(t)=\int_{\Omega}\int_0^t g(t-s)\big|\ \nabla u(x,t)-\nabla u(x,s)\ \big|^2\,ds\,dx.$$

3.2 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 3.1 *Let* (u,v,z) *be the solution of* (3.4). *Then the energy functional, defined by* (3.6), *satisfies*

$$E'(t)\leq -m_0\Big(\int_{\Omega}|u_t|^2dx+\int_{\Omega}z^2(x,1,t)dx\Big)+\frac{1}{2}(g'\circ\nabla u)(t)$$
$$-\frac{1}{2}g(t)\int_{\Omega}|\nabla u|^2dx-\delta\int_{\Omega}|\nabla v_t|^2dx\leq 0,\quad \forall t\geq 0, \tag{3.7}$$

for some constant m_0, where $m_0>0$ if $|\mu_2|<\mu_1$ and $m_0=0$ if $\mu_1=|\mu_2|$.

Proof. A multiplication of the first and the second equation in (3.4) by u_t and v_t respectively, and integration over Ω, using integration by parts and the boundary conditions, yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_t|^2 dx + \int_\Omega v_t^2 dx + \Big(\mu - \int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2 dx + \kappa\int_\Omega |\nabla v|^2 dx \\
+ (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx + (g\circ\nabla u)(t)\Big\} \\
= \frac{1}{2}(g'\circ\nabla u)(t) - \delta\int_\Omega |\nabla v_t|^2 dx - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx \\
- \mu_1\int_\Omega |u_t|^2 dx - \mu_2\int_\Omega u_t\cdot z(x,1,t)dx.
\end{aligned} \tag{3.8}
$$

Now, multiplying the third equation in (3.4) by ξz and integrating over $\Omega\times(0,1)$, we obtain

$$
\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx = -\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi}{2\tau}\int_\Omega |u_t|^2 dx. \tag{3.9}
$$

A combination of (3.8) and (3.9), leads to

$$
\begin{aligned}
E'(t) = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta\int_\Omega |\nabla v_t|^2 dx - \big(\mu_1 - \frac{\xi}{2\tau}\big)\int_\Omega |u_t|^2 dx \\
- \mu_2\int_\Omega u_t\cdot z(x,1,t)dx - \frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx.
\end{aligned}
$$

Then, by Young's inequality, we have

$$
\begin{aligned}
E'(t) \le \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta\int_\Omega |\nabla v_t|^2 dx \\
- \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega |u_t|^2 dx - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega z^2(x,1,t)dx.
\end{aligned}
$$

Consequently, using (3.5), estimate (3.7) follows. ∎

Lemma 3.2 *Suppose that* (A1) *and* (A2) *hold and let* (u, v, z) *be the solution of* (3.4)*. Then the functional*

$$F_1(t) := \int_\Omega u_t \cdot u dx,$$

satisfies, for some positive constant m_1*, the following estimate:*

$$\begin{aligned} F_1'(t) \leq -m_1\Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega (\operatorname{div} u)^2 dx\Big) \\ + c\Big(\int_\Omega |u_t|^2 dx + \int_\Omega v_t^2 dx + \int_\Omega z^2(x,1,t)dx + (g \circ \nabla u)(t)\Big). \end{aligned} \tag{3.10}$$

Proof. Direct computations using the first equation in (3.4), yield

$$\begin{aligned} F_1'(t) = \int_\Omega |u_t|^2 dx - \mu \int_\Omega |\nabla u|^2 dx - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx + \beta \int_\Omega v_t \cdot \operatorname{div} u dx \\ + \int_\Omega \nabla u \cdot \int_0^t g(t-s)\nabla u(s)\, ds\, dx - \mu_1 \int_\Omega u \cdot u_t dx - \mu_2 \int_\Omega z(x,1,t) \cdot u\, dx. \end{aligned}$$

Using Young's and Poincaré's inequalities, for $\delta_1 > 0$, we have

$$\begin{aligned} F_1'(t) \leq -\Big(\frac{\mu}{2} - \delta_1(\mu_1 + |\mu_2|)\Big)\int_\Omega |\nabla u|^2 dx + \frac{1}{2\mu}\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s) ds\Big)^2 dx \\ + \big(1 + \frac{c\mu_1}{4\delta_1}\big)\int_\Omega |u_t|^2 dx - (\mu + \lambda - \delta_1)\int_\Omega (\operatorname{div} u)^2 dx + \frac{1}{4\delta_1}\int_\Omega v_t^2 dx \\ + \frac{c|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx. \end{aligned} \tag{3.11}$$

The second term in the right-hand side of (3.11) is estimated as follows:

$$\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)|ds\Big)^2 dx$$
$$\leq \int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\Big)^2 dx$$
$$= \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx + \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)^2 dx$$
$$+ 2\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)\Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)dx.$$

A simple calculation, using Cauchy-Schwarz and Young's inequalities, for $\eta > 0$, gives

$$\begin{aligned}&\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)|ds\Big)^2 dx \\ &\qquad \leq (\mu-l)^2(1+\eta)\int_\Omega |\nabla u|^2 dx + (\mu-l)\big(1+\frac{1}{\eta}\big)(g\circ\nabla u)(t).\end{aligned} \tag{3.12}$$

By inserting (3.12) into (3.11) and choosing $\eta = \frac{l}{\mu-l}$, we arrive at

$$\begin{aligned}F_1'(t) \leq& \big(1+\frac{c\mu_1}{4\delta_1}\big)\int_\Omega |u_t|^2 dx - \big(\frac{l}{2}-\delta_1(\mu_1+|\mu_2|)\big)\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta_1}\int_\Omega v_t^2 dx \\ &- (\mu+\lambda-\delta_1)\int_\Omega (\operatorname{div} u)^2 dx + \frac{c|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx + \frac{(\mu-l)}{2l}(g\circ\nabla u)(t).\end{aligned}$$

By taking δ_1 small enough, (3.10) follows. ∎

Lemma 3.3 *let (u, v, z) be the solution of (3.4). Then the functional*

$$F_2(t) := \int_\Omega v_t v dx + \beta\int_\Omega v \operatorname{div} u dx + \frac{\delta}{2}\int_\Omega |\nabla v|^2 dx,$$

satisfies, for any positive constant δ_2, the following estimate:

$$F_2'(t) \le -\kappa \int_\Omega |\nabla v|^2 dx + c\big(1 + \frac{1}{\delta_2}\big) \int_\Omega v_t^2 dx + c\delta_2 \int_\Omega (\operatorname{div} u)^2 dx. \tag{3.13}$$

Proof. Taking the derivative of $F_2(t)$ and using the second equation in (3.4), it follows that

$$F_2'(t) = \int_\Omega v_t^2 dx + \kappa \int_\Omega v\Delta v dx + \delta \int_\Omega v\Delta v_t dx + \beta \int_\Omega v_t \operatorname{div} u dx + \delta \int_\Omega \nabla v \cdot \nabla v_t dx.$$

Use of Green's formula and the boundary conditions lead to

$$F_2'(t) = \int_\Omega v_t^2 dx - \kappa \int_\Omega |\nabla v|^2 dx + \beta \int_\Omega v_t \operatorname{div} u dx.$$

By exploiting Young's inequality for $\delta_2 > 0$, estimate (3.13) is established. ∎

As in [82], we state and proof the following lemma:

Lemma 3.4 *let (u, v, z) be the solution of (3.4). Then the functional*

$$F_3(t) := \tau \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x, \rho, t) d\rho dx,$$

satisfies, for some positive constant m_2, the following estimate:

$$F_3'(t) \le -m_2\Big(\int_\Omega z^2(x, 1, t)dx + \tau \int_\Omega \int_0^1 z^2(x, \rho, t)d\rho dx\Big) + \int_\Omega |u_t|^2 dx. \tag{3.14}$$

Proof. By differentiating F_3 and using the third equation in (3.4), we obtain

$$
\begin{aligned}
F_3'(t) &= -2\int_\Omega\int_0^1 e^{-\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t)\,d\rho dx \\
&= -\frac{d}{d\rho}\int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\,d\rho dx - \tau\int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\,d\rho dx \\
&= -\int_\Omega [e^{-\tau} z^2(x,1,t) - z^2(x,0,t)]dx - \tau\int_\Omega\int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\,d\rho dx.
\end{aligned}
$$

By using the fact that $z(x,0,t) = u_t(x,t)$ and $e^{-\tau} \leq e^{-\tau\rho} \leq 1,\ \forall\rho \in [0,1]$, we get

$$
F_3'(t) \leq -e^{-\tau}\left[\int_\Omega z^2(x,1,t)dx + \tau\int_\Omega\int_0^1 z^2(x,\rho,t)\,d\rho dx\right] + \int_\Omega |u_t|^2 dx.
$$

Setting $m_2 = e^{-\tau}$ yields (3.14). ∎

Lemma 3.5 *Suppose that* (A1) *and* (A2) *hold and let* (u, v, z) *be the solution of* (3.4)*. Then, for* $\mu_1 = |\mu_2|$ *and for any* $t_0 > 0$*, the functional*

$$F_4(t) := -\int_\Omega u_t \cdot \int_0^t g(t-s)(u(t)-u(s))\, ds\, dx,$$

satisfies, for some positive constant m_3 *and for any positive constants* δ_3*,* δ_4*,* δ_5*, the following estimate:*

$$\begin{aligned} F_4'(t) \leq & -m_3 \int_\Omega |u_t|^2 dx + c \int_\Omega |\nabla v_t|^2 dx + \delta_3 c \int_\Omega |\nabla u|^2 dx \\ & + c\delta_4 \int_\Omega (\operatorname{div} u)^2 dx + C_\delta (g \circ \nabla u)(t) + c\delta_5 \int_\Omega z^2(x,1,t) dx \\ & - c(g' \circ \nabla u)(t), \quad \forall t \geq t_0 > 0, \end{aligned} \tag{3.15}$$

where $C_\delta = c\left(1 + \delta_3 + \frac{1}{\delta_3} + \frac{1}{\delta_4} + \frac{1}{\delta_5}\right)$

Proof. Differentiating F_4, using (3.4) and integrating by parts together with the boundary conditions, yield

$$
\begin{aligned}
F_4'(t) &= \mu \int_\Omega \nabla u \cdot \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \Big) dx \\
&\quad + (\mu + \lambda) \int_\Omega (\operatorname{div} u) \Big(\int_0^t g(t-s)(\operatorname{div} u(s) - \operatorname{div} u(t)) ds \Big) dx \\
&\quad - \beta \int_\Omega \nabla v_t \cdot \Big(\int_0^t g(t-s)(u(s) - u(t)) ds \Big) dx \\
&\quad - \int_\Omega \Big(\int_0^t g(t-s) \nabla u(s) ds \Big) \cdot \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \Big) dx \qquad (3.16) \\
&\quad + \mu_1 \int_\Omega u_t \cdot \int_0^t g(t-s)(u(s) - u(t)) ds dx - \int_0^t g(s) ds \int_\Omega |u_t|^2 dx \\
&\quad + \mu_2 \int_\Omega z(x,1,t) \cdot \int_0^t g(t-s)(u(s) - u(t)) ds dx \\
&\quad - \int_\Omega u_t \cdot \int_0^t g'(t-s)(u(s) - u(t))\, ds\, dx.
\end{aligned}
$$

Now, we estimate the terms in the right hand side of (3.16) using Young's, Cauchy-Schwarz, and Poincaré's inequalities. So, for $\delta_3, \delta_4, \delta_5, \delta_6 > 0$, we obtain

$$
\begin{aligned}
I_1 &= \int_\Omega \nabla u \cdot \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \Big) dx \\
&\leq \delta_3 \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta_3} \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| ds \Big)^2 dx \\
&\leq \delta_3 \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta_3} \int_0^t g(s) ds \int_\Omega \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|^2 ds dx \qquad (3.17) \\
&\leq \delta_3 \int_\Omega |\nabla u|^2 dx + \frac{\mu - l}{4\delta_3} (g \circ \nabla u)(t),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_\Omega (\operatorname{div} u)\Big(\int_0^t g(t-s)(\operatorname{div} u(s) - \operatorname{div} u(t))ds\Big)dx \\
&\le \delta_4 \int_\Omega (\operatorname{div} u)^2 dx + \frac{1}{4\delta_4}\int_\Omega \Big(\int_0^t g(t-s)\left(\operatorname{div} u(s) - \operatorname{div} u(t)\right) ds\Big)^2 dx \\
&\le \delta_4 \int_\Omega (\operatorname{div} u)^2 dx + \frac{\mu - l}{4\delta_4}\int_\Omega \int_0^t g(t-s)(\operatorname{div} u(s) - \operatorname{div} u(t))^2 \, ds\, dx \\
&\le \delta_4 \int_\Omega (\operatorname{div} u)^2 dx + \frac{\mu - l}{2\delta_4}(g \circ \nabla u)(t),
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
I_3 &= -\int_\Omega \nabla v_t \cdot \Big(\int_0^t g(t-s)(u(s) - u(t))ds\Big)dx \\
&\le \frac{1}{2}\int_\Omega |\nabla v_t|^2 dx + \frac{c(\mu - l)}{2}(g \circ \nabla u)(t),
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
I_4 &= -\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s)ds\Big) \cdot \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds\Big)dx \\
&\le \delta_3 \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)|ds\Big)^2 dx + \frac{1}{4\delta_3}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx \\
&\le 2(\mu - l)^2 \delta_3 \int_\Omega |\nabla u|^2 dx + (\mu - l)\left(2\delta_3 + \frac{1}{4\delta_3}\right)(g \circ \nabla u)(t),
\end{aligned} \tag{3.20}
$$

$$
I_5 = \int_\Omega u_t \cdot \int_0^t g(t-s)(u(s) - u(t))dsdx \le \delta_6 \int_\Omega |u_t|^2 dx + \frac{c(\mu - l)}{4\delta_6}(g \circ \nabla u)(t), \tag{3.21}
$$

$$
\begin{aligned}
I_6 &= \int_\Omega z(x,1,t) \cdot \int_0^t g(t-s)(u(s) - u(t))dsdx \\
&\le \delta_5 \int_\Omega z^2(x,1,t)dx + \frac{c(\mu - l)}{4\delta_5}(g \circ \nabla u)(t),
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
I_7 &= -\int_\Omega u_t \cdot \int_0^t g'(t-s)(u(s) - u(t))\, ds\, dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_6}\int_\Omega \Big(\int_0^t g'(t-s)(u(s) - u(t))ds\Big)^2 dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_6}\int_\Omega \Big(\int_0^t -g'(s)ds\Big)\Big(\int_0^t -g'(t-s)|(u(s) - u(t))|^2 ds\Big)dx \\
&\le \delta_6 \int_\Omega |u_t|^2 dx - \frac{cg(0)}{4\delta_6}(g' \circ \nabla u)(t).
\end{aligned} \tag{3.23}
$$

Since the function g is positive, continuous and $g(0) > 0$, then, for any $t \geq t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0. \tag{3.24}$$

A combination of (3.16)−(3.24) and bearing in mind that $\mu_1 = |\mu_2|$ lead to

$$\begin{aligned}
F_4'(t) \leq &-[g_0 - \delta_6(1+\mu_1)] \int_\Omega |u_t|^2 dx + \delta_5 \mu_1 \int_\Omega z^2(x,1,t)dx - \frac{cg(0)}{4\delta_6}(g' o \nabla u)(t) \\
&+ \frac{\beta}{2} \int_\Omega |\nabla v_t|^2 dx + \delta_3[\mu + 2(\mu - l)^2] \int_\Omega |\nabla u|^2 dx + \delta_4(\mu+\lambda) \int_\Omega (\operatorname{div} u)^2 dx \\
&+ (\mu - l)\Big[\frac{\mu+1}{4\delta_3} + \frac{\mu+\lambda}{2\delta_4} + 2\delta_3 + \frac{c\mu_1}{4}\Big(\frac{1}{\delta_5} + \frac{1}{\delta_6}\Big) + \frac{c\beta}{2}\Big](g \circ \nabla u)(t),
\end{aligned}$$

for all $t \geq t_0$. Next, we choose δ_6 small enough to obtain (3.15). ∎

3.3 Asymptotic Stability

This section is divided into two parts. In the first part, we discuss the case where $|\mu_2| < \mu_1$, and in the second part, we discuss the case where $\mu_1 = |\mu_2|$.

3.3.1 General Decay Result for $|\mu_2| < \mu_1$

For $\varepsilon > 0$ to be chosen appropriately later, we let

$$\mathcal{L}(t) := E(t) + \varepsilon F_1(t) + \varepsilon F_2(t) + \varepsilon F_3(t). \tag{3.25}$$

Lemma 3.6 *There exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t), \quad \forall t \ge 0, \tag{3.26}$$

for ε small enough.

Proof. Let $\mathcal{G}(t) = \varepsilon F_1(t) + \varepsilon F_2(t) + \varepsilon F_3(t)$. By using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}
|\mathcal{G}(t)| \le & \frac{\varepsilon}{2}\int_\Omega \left(|u_t|^2 + v_t^2 + c|\nabla u|^2 + (c(1+\beta)+\delta)\,|\nabla v|^2 + (\operatorname{div} u)^2\right) dx \\
& + \varepsilon\tau \int_\Omega \int_0^1 z^2(x,\rho,t)\,d\rho dx \\
& \le \varepsilon c E(t).
\end{aligned}$$

Consequently, by using (3.25), we obtain $|\mathcal{L}(t) - E(t)| \le \varepsilon c E(t)$. In other words,

$$(1-\varepsilon c)E(t) \le \mathcal{L}(t) \le (1+\varepsilon c)E(t).$$

By choosing ε small enough, (3.26) follows. ∎

Theorem 3.1 *Let (u, v, z) be the solution of* (3.4). *Assume $|\mu_2| < \mu_1$ and* (A1), (A2) *hold. Then, there exist two positive constants c_0 and c_1 such that the energy functional given by* (3.6) *satisfies*

$$E(t) \le c_0 e^{-c_1 \int_0^t \eta(s)ds}, \quad \forall t \ge 0. \tag{3.27}$$

Proof. By differentiating (3.25) and using (3.7), (3.10), (3.13), (3.14) and Poincaré's inequality, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq &-[m_0 - \varepsilon c]\int_\Omega |u_t|^2 dx - \varepsilon m_1 \int_\Omega |\nabla u|^2 dx - \varepsilon\kappa \int_\Omega |\nabla v|^2 dx \\
&- \varepsilon\,[m_1 - c\delta_2]\int_\Omega (\operatorname{div} u)^2 dx - \varepsilon m_2 \tau \int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx \\
&+ \varepsilon c(g \circ \nabla u)(t) - \big[\delta - \varepsilon c\big(1 + \frac{1}{\delta_2}\big)\big]\int_\Omega |\nabla v_t|^2 dx \\
&- [(m_0 - \varepsilon c) + \varepsilon m_2]\int_\Omega z^2(x,1,t)dx.
\end{aligned}$$

At this point, we choose δ_2 small enough such that $(m_1 - c\delta_2) > 0$. Next, by picking

$$0 < \varepsilon < \min\left(\frac{m_0}{c},\ \frac{\delta}{c(1+\frac{1}{\delta_2})}\right),$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq k_1(g \circ \nabla u)(t) - k_2\Big\{&\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \\
&+ \int_\Omega (\operatorname{div} u)^2 dx + \int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx + \int_\Omega |\nabla v_t|^2 dx\Big\},
\end{aligned}$$

for some positive constants k_1 and k_2. Then, using Poincaré's inequality again and (3.6), we get

$$\mathcal{L}'(t) \leq -k_0 E(t) + k_3(g \circ \nabla u)(t), \quad \forall t \geq 0, \tag{3.28}$$

for some positive constants k_0 and k_3.

By multiplying (3.28) by $\eta(t)$ and using (A2) and (3.7), we arrive at

$$\eta(t)\mathcal{L}'(t) \leq -k_0\eta(t)E(t) - 2k_3E'(t), \quad \forall t \geq 0,$$

which can be rewritten as

$$(\eta(t)\mathcal{L}(t) + 2k_3E(t))' - \eta'(t)\mathcal{L}(t) \leq -k_0\eta(t)E(t), \ \forall t \geq 0.$$

Using the fact that $\eta'(t) \leq 0, \forall t \geq 0$, we have

$$(\eta(t)\mathcal{L}(t) + 2k_3E(t))' \leq -k_0\eta(t)E(t), \quad \forall t \geq 0.$$

By exploiting (3.26), it can easily be shown that

$$\mathcal{R}(t) = \eta(t)\mathcal{L}(t) + 2k_3E(t) \sim E(t). \tag{3.29}$$

Consequently, for some positive constant c_1, we obtain

$$\mathcal{R}'(t) \leq -c_1\eta(t)\mathcal{R}(t), \quad \forall t \geq 0. \tag{3.30}$$

A simple integration of (3.30) over $(0,t)$ leads to

$$\mathcal{R}(t) \leq \mathcal{R}(0)e^{-c_1\int_0^t \eta(s)ds}, \quad \forall t \geq 0. \tag{3.31}$$

The conclusion of the theorem follows by combining (3.29) and (3.31). ∎

3.3.2 General Decay Result for $|\mu_2| = \mu_1$

By recalling (3.5), we have $\xi = \tau\mu_1$. Hence, (3.7) takes the form

$$E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx - \delta\int_\Omega |\nabla v_t|^2 dx \leq 0, \quad \forall t \geq 0. \quad (3.32)$$

We then use (3.10), (3.13), and (3.14) with $\mu_1 = |\mu_2|$ and define another Lyapunov functional

$$\tilde{\mathcal{L}}(t) := NE(t) + \varepsilon_1 F_1(t) + F_2(t) + \varepsilon_2 F_3(t) + F_4(t), \quad (3.33)$$

where N, ε_1 and ε_2 are positive real numbers, which will be chosen properly later.

Lemma 3.7 *For N large enough, $\tilde{\mathcal{L}}(t)$ and $E(t)$ satisfy*

$$\alpha_3 E(t) \leq \tilde{\mathcal{L}}(t) \leq \alpha_4 E(t), \quad \forall t \geq 0, \quad (3.34)$$

for two positive constants α_3 and α_4.

Proof. The lemma is established by following the same steps enumerated in the proof of Lemma 3.6 (see page 50). ▮

Theorem 3.2 *Let (u, v, z) be the solution of (3.4). Assume $|\mu_2| = \mu_1$ and (A1), (A2) hold. Then, for any $t_0 > 0$, there exist positive constants c_2 and c_3 such that the energy functional given by (3.6) satisfies*

$$E(t) \leq c_2 e^{-c_3\int_{t_0}^{t}\eta(s)ds}, \quad \forall t \geq t_0. \quad (3.35)$$

Proof. Differentiating $\tilde{\mathcal{L}}$ and using (3.10), (3.13), (3.14), (3.15), (3.32) and Poincaré's inequality, we obtain

$$\begin{aligned}\tilde{\mathcal{L}}'(t) \leq & -[m_3 - \varepsilon_1 c - \varepsilon_2] \int_\Omega |u_t|^2 dx - [\varepsilon_1 m_1 - \delta_3 c] \int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx \\ & - \varepsilon_2 m_2 \tau \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx - [\varepsilon_1 m_1 - c\delta_2 - c\delta_4] \int_\Omega (\operatorname{div} u)^2 dx \\ & - \big[N\delta - c\big(1 + \varepsilon_1 + \frac{1}{\delta_2}\big)\big] \int_\Omega |\nabla v_t|^2 dx + [\frac{N}{2} - c](g' \circ \nabla u)(t) \\ & - [\varepsilon_2 m_2 - c\varepsilon_1 - c\delta_5] \int_\Omega z^2(x,1,t) dx + [\varepsilon_1 c + C_\delta](g \circ \nabla u)(t).\end{aligned}$$

Now, we let

$$\varepsilon_2 = \frac{m3}{2}, \quad \delta_3 = \frac{\varepsilon_1 m_1}{2c}, \quad \delta_4 = \frac{\varepsilon_1 m_1}{2c}.$$

Next, we choose ε_1 small enough so that

$$\tilde{k}_1 := [\frac{m_3}{2} - \varepsilon_1 c] > 0 \quad \text{and} \quad \tilde{k}_2 := [\frac{m_2 m_3}{2} - \varepsilon_1 c] > 0.$$

Once ε_1 is fixed, we then take $\delta_5 = \tilde{k}_2/(2c)$ and choose δ_2 small enough so that

$$\tilde{k}_3 := [\frac{\varepsilon_1 m_1}{2} - c\delta_2] > 0.$$

Finally, we choose N so large that (3.34) remains valid and furthermore,

$$\tilde{k}_4 := \big[N\delta - -c\big(1 + \varepsilon_1 + \frac{1}{\delta_2}\big)\big] > 0 \quad \text{and} \quad [\frac{N}{2} - c] > 0.$$

Hence, we arrive at

$$\begin{aligned}\tilde{\mathcal{L}}'(t) \leq &-\tilde{k}_1 \int_\Omega |u_t|^2 dx - \frac{\varepsilon_1 m_1}{2} \int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx \\ &- \tilde{k}_4 \int_\Omega |\nabla v_t|^2 dx - \tilde{k}_3 \int_\Omega (\operatorname{div} u)^2 dx + \tilde{k}_5 (g \circ \nabla u)(t) \\ &- \frac{m_2 m_3 \tau}{2} \int_\Omega \int_0^1 z^2(x, \rho, t) d\rho dx.\end{aligned}$$

Using Poincaré's inequality and (3.6), we obtain

$$\tilde{\mathcal{L}}'(t) \leq -\tilde{k}_0 E(t) + \tilde{k}_5 (g \circ \nabla u)(t), \quad \forall t \geq t_0, \tag{3.36}$$

where $\tilde{k}_0$ and $\tilde{k}_5$ are two positive constants.

By multiplying (3.36) by $\eta(t)$ and using (A2) and (3.32), we obtain

$$\eta(t)\tilde{\mathcal{L}}'(t) \leq -\tilde{k}_0 \eta(t) E(t) - 2\tilde{k}_5 E'(t), \quad \forall t \geq t_0,$$

which implies that

$$\Big(\eta(t)\tilde{\mathcal{L}}(t) + 2\tilde{k}_5 E(t)\Big)' \leq -\tilde{k}_0 \eta(t) E(t), \quad \forall t \geq t_0.$$

If we set

$$\tilde{\mathcal{R}}(t) = \eta(t)\tilde{\mathcal{L}}(t) + 2\tilde{k}_5 E(t) \sim E(t) \tag{3.37}$$

and follow the same steps as in Theorem 3.1 (see page 53-53), we arrive at

$$\tilde{\mathcal{R}}(t) \leq \tilde{\mathcal{R}}(t_0) e^{-c_3 \int_{t_0}^{t} \eta(s) ds}, \quad \forall t \geq t_0. \tag{3.38}$$

Consequently, (3.35) is established by virtue of (3.37) and (3.38). ▌

Remark 3.2 *Estimate* (3.35) *also holds for $t \in [0, t_0]$ by the continuity and boundedness of E and η. In other words, since $E(t) \leq E(t_0) \leq E(0)$, $\forall t \geq t_0 > 0$, we get, for some $\tilde{c}_2$,*

$$E(t) \leq \tilde{c}_2 E(0) e^{c_3 \int_0^{t_0} \eta(s) ds} e^{-c_3 \int_0^{t} \eta(s) ds}, \quad \forall t \geq t_0 > 0.$$

Consequently, by taking $c_2 = \tilde{c}_2 E(0) e^{c_3 \int_0^{t_0} \eta(s) ds}$ we obtain the estimate, for all $t \geq 0$.

Now, we give some examples to illustrate the energy decay rates obtained by Theorem 3.1, which is also valid for Theorem 3.2. We consider the three examples under Remark 3.1 with the same assumptions on a and b as stated before.

1. If $g(t) = ae^{-bt}$, then

$$E(t) \leq c_0 e^{-bc_1 t}, \quad \forall t \geq 0.$$

2. If $g(t) = \frac{a}{(1+t)^{b+1}}$, then

$$E(t) \leq \frac{c_0}{(1+t)^{(b+1)c_1}}, \quad \forall t \geq 0.$$

3. If $g(t) = \frac{a}{(e+t)[ln(e+t)]^{b+1}}$, then

$$E(t) \leq \frac{c_0 e^{c_1}}{\{(e+t)[ln(e+t)]^{b+1}\}^{c_1}}, \quad \forall t \geq 0.$$

Remark 3.3 Our result extends the result obtained by Kirane and Said-Houari [44] for viscoelastic wave equation with a delay to thermoelasticity of type III with delay. Furthemore, in contrast to [44], we do not require that μ_2 be positive.

CHAPTER 4

THERMOELASTICITY OF TYPE III WITH A DELAY TERM AND INFINITE MEMORY

In this chapter, we investigate the asymptotic behavior of a thermoelastic system of type III with infinite memory and internal delay. Under suitable assumptions on the weight of the delay term, we prove the well-posedness of the system. Furthermore, for a wide class of relaxation functions, we use the multiplier method and a convexity argument to establish a general stability result of the system. To this end, we consider the following thermoviscoelastic problem:

$$
\begin{cases}
u_{tt}(x,t) - \mu\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\
\quad + \displaystyle\int_0^\infty g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \text{ in } \Omega\times(0,\infty), \\
\theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta \operatorname{div} u_{tt}(x,t), = 0 \qquad \text{in } \Omega\times(0,\infty), \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x) \quad \text{in } \Omega, \\
u(x,-t) = f_0(x,t) \qquad \text{in } \Omega\times[0,\infty), \\
u_t(x,-t) = f_1(x,t) \qquad \text{in } \Omega\times(0,\tau), \\
u(x,t) = \theta(x,t) = 0 \qquad \text{on } \partial\Omega\times[0,\infty),
\end{cases}
\tag{4.1}
$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a boundary $\partial\Omega$ of class C^2, $u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta(x,t)$ is the difference temperature, the relaxation function g is positive and decreasing, the coefficients $\mu, \lambda, \beta, \mu_1, \kappa, \delta$ are positive constants, μ_2 is a real number and $\tau > 0$ represents the time delay. This is a (type III) thermoelastic system with the presence of an infinite memory and a constant internal delay supplemented by initial data $u_0, u_1, \theta_0, \theta_1$ and history functions f_0 and f_1.

4.1 Introduction

Regarding infinite history problems, Appleby *et al* [4] investigated a linear integro-differential equation and established exponential decay of strong solutions in a

Hilbert space. Pata [92] discussed the decay properties of the semigroup generated by a linear integro-differential equation in a Hilbert space, which is an abstract version of

$$u_{tt} - \Delta u + \int_0^{\infty} \mu(s)\Delta u(t-s)ds = 0 \qquad \text{in } \Omega \times (0,\infty),$$

and established necessary and sufficient conditions for the exponential stability. Muñoz Rivera and Fernández Sare [77] examined, in $(0,L) \times (0,\infty)$, the following system

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \displaystyle\int_0^{\infty} g(s)\psi_{xx}(x,t-s)ds + k(\varphi_x + \psi) = 0, \end{cases} \tag{4.2}$$

where g is a positive twice differentiable function satisfying, for some constants $k_0, k_1, k_2 > 0$

$$-k_0 g(t) \leq g'(t) \leq -k_1 g(t) \text{ and } |g''(t)| \leq k_2 g(t), \qquad \forall t \in \mathbb{R}_+ \tag{4.3}$$

and

$$b - \int_0^{+\infty} g(s)ds > 0$$

and showed that the dissipation given by the memory term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal

$\left(\dfrac{k}{\rho_1} = \dfrac{b}{\rho_2}\right)$. They also proved that the energy of regular solutions decays polynomially for the case of different wave speeds $\left(\dfrac{k}{\rho_1} \neq \dfrac{b}{\rho_2}\right)$. Messaoudi and Said-Houari [66] discussed (4.2) when g is decaying polynomially and established some stability results under weaker conditions than (4.3).

Recently, Guesmia [28] considered

$$u'' + Au - \int_0^\infty g(s)Bu(t-s)ds = 0, \qquad \forall t > 0$$

and introduced a new ingenuous approach which allows a larger class of infinite history kernels than the one considered in the literature thereby established a more general decay result for a class of hyperbolic problems. Using this approach, Guesmia and Messaoudi [29] later considered

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\,\mathrm{div}(a_1(x)\nabla u(s))ds + \int_0^\infty g_2\,\mathrm{div}(a_2(x)\nabla u(t-s))ds = 0,$$

in a bounded domain and under suitable conditions on a_1 and a_2 and for a wide class of relaxation functions g_1 and g_2 that are not necessarily decaying polynomially or exponentially, and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. Using this same approach, Guesmia *et al.* [30] revisited (4.2), as well as different kind of coupled Timoshenko-heat systems and established general decay results from which the results in [66] and [77] are only particular cases. More recently, Guesmia and Messaoudi [33] considered the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + \displaystyle\int_0^\infty g(s)\psi_{xx}(t-s)ds = 0, \end{cases}$$

where the relaxation function g satisfies the same conditions and hypothesis imposed for the finite memory case[1]. They established some general decay results for the cases of equal and nonequal speeds of wave propagation. Their method of proof requires no convex function properties or the generalized Young inequality as required in [28]. For more results on infinite history, we refer readers to [31, 32, 57, 105] and the references therein.

The rest of the chapter is organized as follows. In section 4.2, we introduce some transformations and assumptions needed in this chapter. In section 4.3, we use the semi-group theory to prove the well-posedness of the problem. Some technical lemmas and the statement with proof of our main results will be given in section 4.4 and section 4.5, respectively.

4.2 Assumptions and Transformations

In this section, we present some materials needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. For the relaxation function g, we assume the following:

[1]See assumptions (A1) and (A2) in Chapter 3, page 37.

(A) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \quad \mu - \int_0^\infty g(s)ds = l > 0 \tag{4.4}$$

and there exists a positive constant γ and a strictly increasing and strictly convex positive function $G \in C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$, with $G(0) = G'(0) = 0$ and $\lim\limits_{t\to+\infty} G'(t) = +\infty$ such that

$$g'(t) \leq -\gamma g(t),\ \forall t \geq 0, \tag{4.5}$$

or

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}ds + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{4.6}$$

Remark 4.1 To the best of our knowledge, inequality (4.6) was introduced first in [28] to study the general stability of an abstract system with infinite memory.

In addition to the transformation in section 3.1 (see page 38-39), we introduce as in Dafermos [15], the relative history of u

$$\eta^t(x,s) = u(x,t) - u(x,t-s),\ \text{in } \Omega \times (0,\infty) \times (0,\infty).$$

Simple differentiation shows that η^t satisfies

$$\eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t), \quad \text{in } \Omega \times (0,\infty) \times (0,\infty)$$

Consequently, problem (4.1) takes the form

$$\begin{cases} u_{tt}(x,t) - l\Delta u(x,t) - (\mu+\lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla v_t(x,t) \\ \quad - \displaystyle\int_0^\infty g(s)\Delta\eta^t(x,s)ds + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0 \text{ in } \Omega\times(0,\infty), \\ v_{tt}(x,t) - \kappa\Delta v(x,t) - \delta\Delta v_t(x,t) + \beta \operatorname{div} u_t(x,t) = 0 \quad \text{in } \Omega\times(0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega\times(0,1)\times(0,\infty), \\ \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t) \quad \text{in } \Omega\times(0,\infty)\times(0,\infty), \\ z(x,0,t) = u_t(x,t) \quad \text{in } \Omega\times(0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), v(x,0) = v_0(x), v_t(x,0) = v_1(x) \text{ in } \Omega, \\ \eta^0(x,s) = \eta_0(x,s) = f_0(x,0) - f_0(x,s),\ \eta^t(x,0) = 0 \quad \text{in } \Omega\times(0,\infty), \\ z(x,\rho,0) = f_1(x,\tau\rho) \quad \text{in } \Omega\times(0,1), \\ u(x,t) = v(x,t) = \eta^t(x,s) = 0 \quad \text{on } \partial\Omega\times(0,\infty)\times(0,\infty). \end{cases} \tag{4.7}$$

Thus, we shall consider problem (4.7) instead of (4.1).

As in Chapter 3, page 40, we assume that $|\mu_2| \leq \mu_1$ and let ξ be a positive constant satisfying

$$\begin{cases} \tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), & \text{if } |\mu_2| < \mu_1, \\ \xi = \tau\mu_1, & \text{if } |\mu_2| = \mu_1. \end{cases} \tag{4.8}$$

The energy associated with problem (4.7) is given by

$$
\begin{aligned}
E(t) = E(t,u,v,z,\eta^t) = &\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega v_t^2 dx + \frac{l}{2}\int_\Omega |\nabla u|^2 dx + \frac{\kappa}{2}\int_\Omega |\nabla v|^2 dx \\
&+ \frac{(\mu+\lambda)}{2}\int_\Omega (\operatorname{div} u)^2 dx + \frac{1}{2}(g\circ\nabla\eta^t)(t) + \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx,
\end{aligned} \tag{4.9}
$$

where

$$
(g\circ\nabla\eta^t)(t) = \int_0^\infty g(s)\int_\Omega \mid \nabla\eta^t(x,s)\mid^2 dxds.
$$

4.3 The Well-posedness of the Problem

In this section, we give the existence and uniqueness result for problem (4.7) using semi-group theory. We consider the following Hilbert space:

$$
\mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times H_0^1(\Omega) \times L^2(\Omega) \times L^2((0,1),\ (L^2(\Omega))^n) \times L_g^2(\mathbb{R}_+,\ (H_0^1(\Omega))^n),
$$

where $L_g^2(\mathbb{R}_+,\ (H_0^1(\Omega))^n)$ denotes the Hilbert space of $(H_0^1(\Omega))^n$-valued functions on $\mathbb{R}_+$ equipped with the norm

$$
\| w \|^2_{L_g^2(\mathbb{R}_+,\ (H_0^1(\Omega))^n)} = \int_0^\infty g(s) \| w(s) \|^2_{(H_0^1(\Omega))^n} ds.
$$

and

$$
L^2((0,1),\ (L^2(\Omega))^n) = \left\{ w\colon (0,1) \to (L^2(\Omega))^n / \int_0^1 \| w(s) \|^2_{(L^2(\Omega))^n} ds < \infty \right\}.
$$

We equip $\mathcal{H}$ with the inner product

$$
\begin{aligned}
(\Phi,\tilde{\Phi})_{\mathcal{H}} = {} & l\int_\Omega \nabla u\cdot\nabla\tilde{u}dx + (\mu+\lambda)\int_\Omega \operatorname{div} u\cdot\operatorname{div}\tilde{u}dx + \int_\Omega \varphi\cdot\tilde{\varphi}dx + \int_\Omega \psi\tilde{\psi}dx \\
& + \kappa\int_\Omega \nabla v\cdot\nabla\tilde{v}dx + \int_\Omega\int_0^\infty g(s)\nabla w\cdot\nabla\tilde{w}dsdx + \xi\int_\Omega\int_0^1 z\cdot\tilde{z}d\rho dx,
\end{aligned}
\tag{4.10}
$$

for $\Phi = (u,\varphi,v,\psi,z,w)^T$, $\tilde{\Phi} = (\tilde{u},\tilde{\varphi},\tilde{v},\tilde{\psi},\tilde{z},\tilde{w})^T \in \mathcal{H}$. It is easy to check that $\mathcal{H}$, with respect to (4.10), forms a Hilbert space.

With $\Phi = (u,\varphi,v,\psi,z,w)^T$, where $\varphi = u_t, \psi = v_t$ and $w = \eta^t$, system (4.7) can be rewritten in the following form:

$$
\begin{cases}
\Phi'(t) + \mathcal{A}\Phi(t) = 0, \qquad t > 0, \\
\Phi(0) = \Phi_0 = (u_0,u_1,v_0,v_1,f_1,\eta_0)^T,
\end{cases}
\tag{4.11}
$$

where the operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is defined by

$$
\mathcal{A}\Phi = \begin{pmatrix}
-\varphi \\
-l\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\psi - \int_0^\infty g(s)\Delta w(x,s)ds + \mu_1\varphi + \mu_2 z(.,1) \\
-\psi \\
-\kappa\Delta v - \delta\Delta\psi + \beta\operatorname{div}\varphi \\
\frac{1}{\tau}z_\rho \\
-\varphi + w_s
\end{pmatrix}
$$

with domain

$$
D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} \ : \ u, \varphi \in \left(H_0^1(\Omega)\right)^n, \qquad lu + \int_0^\infty g(s) w(s) ds \in \left(H^2(\Omega)\right)^n, \\ \\ v, \psi \in H_0^1(\Omega), \qquad \kappa v + \delta \psi \ \in H^2(\Omega), \\ \\ z, z_\rho \in L^2\left((0,1), \left(L^2(\Omega)\right)^n\right), \qquad z(0) = \varphi(x), \\ \\ w, w_s \ \in L_g^2\left(\mathbb{R}_+, \ \left(H_0^1(\Omega)\right)^n\right), \qquad w(0) = 0 \end{array} \right\}.
$$

We have the following existence and uniqueness result:

Theorem 4.1 *Assume that* $|\mu_2| \leq \mu_1$ *and that* g *satisfies* (4.4)*, then for any* $\Phi_0 \in \mathcal{H}$*, there exists a unique solution* $\Phi \in C(\mathbb{R}_+, \mathcal{H})$ *of problem* (4.11)*. Moreover, if* $\Phi_0 \in D(\mathcal{A})$*, then* $\Phi \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$*.*

Proof. The result follows from the Hille-Yosida theorem provided we prove that $\mathcal{A}$ is a maximal monotone operator (see [6], Chapter 7). First, we prove that $\mathcal{A}$ is monotone. For any $\Phi \in D(\mathcal{A})$, we have

$$
\begin{aligned}
(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} =& \mu_1 \int_\Omega |\varphi|^2 dx + \mu_2 \int_\Omega z(x,1,t) \cdot \varphi dx + \frac{\xi}{\tau} \int_\Omega \int_0^1 z(x,\rho,t) \cdot z_\rho(x,\rho,t) d\rho dx \\
&+ \delta \int_\Omega |\nabla \psi|^2 dx + \int_\Omega \int_0^\infty g(s) \nabla w(x,s) \cdot \nabla w_s(x,s) ds dx.
\end{aligned} \tag{4.12}
$$

Using Young's inequality, the second term in the right hand side of (4.12) gives

$$
-\mu_2 \int_\Omega z(x,1,t) \cdot \varphi dx \leq \frac{|\mu_2|}{2} \int_\Omega z^2(x,1,t) dx + \frac{|\mu_2|}{2} \int_\Omega |\varphi|^2 dx.
$$

Also, using integration by parts and the fact that $z(x,0) = \varphi(x)$, the third term in the right hand side of (4.12) gives

$$\begin{aligned}\int_\Omega\int_0^1 z(x,\rho,t)\cdot z_\rho(x,\rho,t)d\rho dx &= \frac{1}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}z^2(x,1,t)d\rho dx\\ &= \frac{1}{2}\int_\Omega z^2(x,1,t)dx-\frac{1}{2}\int_\Omega z^2(x,0,t)dx\\ &= \frac{1}{2}\int_\Omega z^2(x,1,t)dx-\frac{1}{2}\int_\Omega |\varphi|^2dx.\end{aligned}$$

Similarly, the last term of (4.12) gives

$$\begin{aligned}\int_\Omega\int_0^\infty g(s)\nabla w(x,s)\cdot\nabla w_s(x,s)dsdx =&\frac{1}{2}\int_\Omega\int_0^\infty g(s)\frac{\partial}{\partial s}|\nabla w(x,s)|^2dsdx\\ =&\frac{1}{2}\int_\Omega\int_0^\infty \frac{\partial}{\partial s}g(s)|\nabla w(x,s)|^2dsdx\\ &-\frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\nabla w(x,s)|^2dsdx\\ =&-\frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\nabla w(x,s)|^2dsdx.\end{aligned}$$

Consequently, (4.12) yields

$$\begin{aligned}(\mathcal{A}\Phi,\Phi)_{\mathcal{H}} \geq \left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega|\varphi|^2dx+\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega|z(x,1,t)|^2dx\\ +\delta\int_\Omega|\nabla\psi|^2dx-\frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\nabla w(x,s)|^2dsdx.\end{aligned}$$

In view of (4.8), we have

$$\begin{aligned}(\mathcal{A}\Phi,\Phi)_{\mathcal{H}} \geq& m_0\int_\Omega\left(|\varphi|^2+|z(x,1,t)|^2\right)dx+\delta\int_\Omega|\nabla\psi|^2dx\\ &-\frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\nabla w(x,s)|^2dsdx,\end{aligned}\tag{4.13}$$

for some constant m_0, where $m_0>0$ if $|\mu_2|<\mu_1$ and $m_0=0$ if $|\mu_2|=\mu_1$.

Since g is nonincreasing, we conclude from (4.13) that $(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \geq 0$, which implies that $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective. Given $\mathcal{G} = (g_1, g_2, g_3, g_4, g_5, g_6)^T \in \mathcal{H}$, we prove that there exists $\Phi \in D(\mathcal{A})$ satisfying

$$(I + \mathcal{A})\,\Phi = \mathcal{G}, \tag{4.14}$$

which is equivalent to the following system:

$$\begin{cases} -\varphi + u = g_1, \\ -l\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \beta\nabla\psi - \displaystyle\int_0^\infty g(s)\Delta w(s)ds + (1+\mu_1)\varphi + \mu_2 z(.,1) = g_2, \\ -\psi + v = g_3, \\ -\kappa\Delta v - \delta\Delta\psi + \beta \operatorname{div}\varphi + \psi = g_4, \\ z_\rho + \tau z = \tau g_5, \\ -\varphi + w_s + w = g_6. \end{cases} \tag{4.15}$$

In what follows, we use $(4.15)_i, i = 1,2,3,4,5,6$, to refer to the ith equation in (4.15).

Suppose u and v are given with the appropriate regularity, then $(4.15)_1$ and $(4.15)_3$ give

$$\varphi = u - g_1 \in (H_0^1(\Omega))^n \tag{4.16}$$

and

$$\psi = v - g_3 \in H_0^1(\Omega), \tag{4.17}$$

respectively.

Use of $(4.15)_5$ together with (4.16) and the fact that $z(x,0)=\varphi(x)$ easily gives

$$z(x,\rho)=e^{-\tau\rho}u(x)-e^{-\tau\rho}g_1+\tau e^{-\tau\rho}\int_0^{\rho}e^{\tau\alpha}g_5(\alpha)d\alpha. \tag{4.18}$$

Similarly, using $(4.15)_6$ and (4.16), we define

$$w(x,s)=(1-e^{-s})(u-g_1)+e^{-s}\int_0^{s}e^{y}g_6(y)dy. \tag{4.19}$$

Using Green's formula, it can easily be shown that equations $(4.15)_2$ and $(4.15)_4$ satisfy the following:

$$\begin{cases} (1+\mu_1)\displaystyle\int_\Omega \varphi\cdot u_1dx+l\int_\Omega \nabla u\cdot\nabla u_1dx+(\mu+\lambda)\int_\Omega \operatorname{div}u\operatorname{div}u_1dx \\ \qquad +\beta\displaystyle\int_\Omega u_1\cdot\nabla\psi dx+\int_\Omega\nabla u_1\cdot\int_0^\infty g(s)\nabla wsdsdx+\mu_2\int_\Omega z(.,1)\cdot u_1dx \\ \qquad\qquad =\displaystyle\int_\Omega g_2\cdot u_1dx,\forall u_1\in(H_0^1(\Omega))^n \\ \displaystyle\int_\Omega \psi v_1dx+\kappa\int_\Omega\nabla v\cdot\nabla v_1dx+\delta\int_\Omega\nabla\psi\cdot\nabla v_1dx+\beta\int_\Omega v_1\operatorname{div}\varphi dx \\ \qquad\qquad =\displaystyle\int_\Omega g_4v_1dx,\forall v_1\in H_0^1(\Omega). \end{cases}$$

Furthermore, by using (4.16) – (4.19), we have the following corresponding weak formulation for $(4.15)_2$ and $(4.15)_4$: Finding $(u,v)\in((H_0^1(\Omega))^n\times H_0^1(\Omega))$ such that for all $(u_1,v_1)\in((H_0^1(\Omega))^n\times H_0^1(\Omega))$ the following holds:

$$B\big((u,v),(u_1,v_1)\big)=F\big(u_1,v_1\big), \tag{4.20}$$

where B is the bilinear form on $(H_0^1(\Omega))^n \times H_0^1(\Omega)$ defined by

$$\begin{aligned}
B\big((u,v),(u_1,v_1)\big) = & \tilde{\mu}\int_\Omega u\cdot u_1 dx + \tilde{l}\int_\Omega \nabla u \cdot \nabla u_1 dx + (\mu+\lambda)\int_\Omega \operatorname{div} u \operatorname{div} u_1 dx \\
& + \beta\int_\Omega u_1\cdot\nabla v dx + \int_\Omega v\cdot v_1 dx + \beta\int_\Omega v_1 \operatorname{div} u dx + (\kappa+\delta)\int_\Omega \nabla v\cdot\nabla v_1 dx,
\end{aligned}$$

F is the linear form on $(H_0^1(\Omega))^n \times H_0^1(\Omega)$ defined by

$$\begin{aligned}
F\big(u_1,v_1\big) = \tilde{\mu}\int_\Omega g_1\cdot u_1 dx &+ \int_\Omega g_2\cdot u_1 dx + \int_\Omega \nabla u_1\cdot\nabla g_1 \int_0^\infty g(s)\left(1-e^{-s}\right) ds dx \\
&+ \beta\int_\Omega u_1\cdot\nabla g_3 dx - \int_\Omega \nabla u_1 \cdot \int_0^\infty g(s)e^{-s}\int_0^s e^y \nabla g_6(x,y) dy ds dx \\
&- \mu_2\tau e^{-\tau}\int_\Omega u_1\cdot\int_0^1 e^{\tau\alpha} g_5(x,\alpha) d\alpha dx + \int_\Omega \left(g_3+g_4\right) v_1 dx \\
&+ \delta\int_\Omega \nabla g_3\cdot\nabla v_1 dx - \beta\int_\Omega g_1\cdot\nabla v_1 dx,
\end{aligned}$$

and

$$\tilde{\mu} = 1+\mu_1+\mu_2 e^{-\tau}, \quad \tilde{l} = l + \int_0^\infty g(s)(1-e^{-s})ds.$$

Let $V = H_0^1(\Omega)$, then for $V^n\times V$ equipped with the norm

$$\begin{aligned}
\|(u,v)\|^2_{V^n\times V} =& \|u\|^2_{(L^2(\Omega))^n} + \|\nabla u\|^2_{(L^2(\Omega))^n} + \|\operatorname{div} u\|^2_{(L^2(\Omega))^n} \\
&+ \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \\
=& \|u\|^2_{V^n} + \|\operatorname{div} u\|^2_{(L^2(\Omega))^n} + \|v\|^2_V,
\end{aligned}$$

we prove that B and F are bounded.

$$
\begin{aligned}
|B\big((u,v),(u_1,v_1)\big)| \le |\tilde{\mu}| \int_\Omega |u||u_1| dx + \tilde{l} \int_\Omega |\nabla u||\nabla u_1| dx + \int_\Omega |v||v_1| dx \\
+ \beta \int_\Omega |\nabla v||u_1| dx + (\mu+\lambda) \int_\Omega |u||\operatorname{div} u_1| dx \\
+ \beta \int_\Omega |\operatorname{div} u||v_1| dx + (\kappa+\delta) \int_\Omega |\nabla v||\nabla v_1| dx.
\end{aligned}
$$

By using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|B\big((u,v),(u_1,v_1)\big)| &\le |\tilde{\mu}| \|u\|_{(L^2(\Omega))^n} \|u_1\|_{(L^2(\Omega))^n} + \tilde{l} \|\nabla u\|_{(L^2(\Omega))^n} \|\nabla u_1\|_{(L^2(\Omega))^n} \\
&\quad + (\mu+\lambda) \|u\|_{(L^2(\Omega))^n} \|\operatorname{div} u_1\|_{(L^2(\Omega))^n} \\
&\quad + \beta \|\nabla v\|_{L^2(\Omega)} \|u_1\|_{(L^2(\Omega))^n} + (\kappa+\delta) \|\nabla v\|_{L^2(\Omega)} \|\nabla v_1\|_{L^2(\Omega)} \\
&\quad + \|v\|_{L^2(\Omega)} \|v_1\|_{L^2(\Omega)} + \beta \|\operatorname{div} u\|_{(L^2(\Omega))^n} \|v_1\|_{L^2(\Omega)} \\
&\le c\|u\|_{V^n} \|u_1\|_{V^n} + c\|v\|_V \|v_1\|_V + c\|v\|_V \|u_1\|_{V^n} \\
&\quad + c(\|u\|_{V^n} + \|\operatorname{div} u\|_{(L^2(\Omega))^n})(\|v_1\|_V + \|\operatorname{div} u_1\|_{(L^2(\Omega))^n}) \\
&\le c \left(\|u\|_{V^n} + \|v\|_V + \|\operatorname{div} u\|_{(L^2(\Omega))^n} \right) \\
&\qquad \left(\|u_1\|_{V^n} + \|v_1\|_V + \|\operatorname{div} u_1\|_{(L^2(\Omega))^n} \right) \\
&\le c\|(u,v)\|_{V^n \times V} \|(u_1,v_1)\|_{V^n \times V}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
|F(u_1,v_1)| \le{}& |\tilde{\mu}|\|g_1\|_{(L^2(\Omega))^n}\|u_1\|_{(L^2(\Omega))^n} + \|g_2\|_{(L^2(\Omega))^n}\|u_1\|_{(L^2(\Omega))^n} \\
&+ \beta\|\nabla g_3\|_{L^2(\Omega)}\|u_1\|_{(L^2(\Omega))^n} + 2(\mu - l)\|\nabla g_1\|_{(L^2(\Omega))^n}\|\nabla u_1\|_{(L^2(\Omega))^n} \\
&+ \tau|\mu_2|\|g_5\|_{L^2((0,1),\ (L^2(\Omega))^n)}\|u_1\|_{(L^2(\Omega))^n} + \beta\|g_1\|_{(L^2(\Omega))^n}\|\nabla v_1\|_{L^2(\Omega)} \\
&+ \|g_3\|_{L^2(\Omega)}\|v_1\|_{L^2(\Omega)} + \|\nabla g_3\|_{L^2(\Omega)}\|\nabla v_1\|_{L^2(\Omega)} \\
&+ \|g_4\|_{L^2(\Omega)}\|v_1\|_{L^2(\Omega)} + \int_\Omega \nabla u_1 \cdot \int_0^\infty g(s)e^{-s}\int_0^s e^y \nabla g_6(x,y)\,dy\,ds\,dx.
\end{aligned}
\tag{4.21}
$$

By using Cauchy-Schwarz inequality, the last term in (4.21) is estimated as follows:

$$
\begin{aligned}
&\int_\Omega \nabla u_1 \cdot \int_0^\infty g(s)e^{-s}\int_0^s e^y \nabla g_6(x,y)\,dy\,ds\,dx \\
&\qquad \le \|\nabla u_1\|_{(L^2(\Omega))^n}\left(\int_\Omega\left(\int_0^\infty g(s)e^{-s}\int_0^s e^y|\nabla g_6(x,y)|\,dy\,ds\right)^2 dx\right)^{1/2} \\
&\qquad \le \|u_1\|_{V^n}\left(\int_\Omega\left(\int_0^\infty\int_0^s g(s)e^{-s}e^y|\nabla g_6(x,y)|\,dy\,ds\right)^2 dx\right)^{1/2} \\
&\qquad = \|u_1\|_{V^n}\left(\int_\Omega\left(\int_0^\infty e^y|\nabla g_6(x,y)|\int_y^\infty g(s)e^{-s}\,ds\,dy\right)^2 dx\right)^{1/2} \\
&\qquad \le \|u_1\|_{V^n}\left(\int_\Omega\left(\int_0^\infty e^y g(y)|\nabla g_6(x,y)|\int_y^\infty e^{-s}\,ds\,dy\right)^2 dx\right)^{1/2} \\
&\qquad = \|u_1\|_{V^n}\left(\int_\Omega\left(\int_0^\infty g(y)|\nabla g_6(x,y)|\,dy\right)^2 dx\right)^{1/2}.
\end{aligned}
\tag{4.22}
$$

Finally, using Cauchy-Schwarz inequality on the last term in (4.22), we get

$$
\int_\Omega \nabla u_1 \cdot \int_0^\infty g(s)e^{-s}\int_0^s e^y \nabla g_6(x,y)\,dy\,ds\,dx \le c\|u_1\|_{V^n}\|g_6\|_{L^2_g(\mathbb{R}_+,\ V^n)}. \tag{4.23}
$$

The combination of (4.21) and (4.23) yields

$$
\begin{aligned}
|F(u_1, v_1)| \le c\Big(& \|g_1\|_{V^n} + \|g_2\|_{(L^2(\Omega))^n} + \|g_3\|_V + \|g_4\|_{L^2(\Omega)} \\
& + \|g_5\|_{L^2((0,\ 1),\ (L^2(\Omega))^n)} + \|g_6\|_{L^2_g(\mathbb{R}_+,\ V^n)}\Big) \left(\|u_1\|_{V^n} + \|v_1\|_V\right) \\
\le{} & c\|(u_1, v_1)\|_{V^n\times V}.
\end{aligned}
$$

Hence, B and F are bounded. On the other hand, from the definition of B, we have

$$
\begin{aligned}
B\big((u,v),(u,v)\big) ={}& \tilde{\mu}\int_\Omega |u|^2 dx + \tilde{l}\int_\Omega |\nabla u|^2 dx + (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx \\
& + \int_\Omega |v|^2 dx + (\kappa+\delta)\int_\Omega |\nabla v|^2 dx \\
& \ge \alpha_0 \|(u,v)\|^2_{V^n\times V},
\end{aligned}
$$

for some $\alpha_0 > 0$, which implies that B is coercive. Consequently, by Lax-Milgram Lemma, there exists a unique

$$
(u, v) \in (H^1_0(\Omega))^n \times H^1_0(\Omega)
$$

satisfying (4.20). By substituting u and v into (4.16)–(4.19), we get

$$
\begin{aligned}
& \varphi \in (H^1_0(\Omega))^n,\ \psi \in H^1_0(\Omega), z \in L^2\left((0,1), \left(L^2(\Omega)\right)^n\right),\ \text{and} \\
& w\ \in L^2_g\left(\mathbb{R}_+,\ \left(H^1_0(\Omega)\right)^n\right).
\end{aligned}
$$

Moreover, by substituting z into $(4.15)_5$, and φ, w into $(4.15)_6$, we obtain

$$z_\rho \in L^2\left((0,1),\left(L^2(\Omega)\right)^n\right) \text{ and } w_s \ \in L^2_g\left(\mathbb{R}_+\left(H^1_0(\Omega)\right)^n\right).$$

It is very clear from (4.18) and (4.19) that $z(x,0)=\varphi(x)$ and $w(x,0)=0$.

Now, if $v_1 \equiv 0 \in H^1_0(\Omega)$, then (4.20) reduces to

$$\begin{aligned}
&\tilde{\mu}\int_\Omega u\cdot u_1 dx + \tilde{l}\int_\Omega \nabla u\cdot\nabla u_1 dx + (\mu+\lambda)\int_\Omega \operatorname{div} u \operatorname{div} u_1 dx + \beta\int_\Omega \nabla v\cdot u_1 dx\\
&\quad = \tilde{\mu}\int_\Omega g_1\cdot u_1 dx + \int_\Omega g_2\cdot u_1 dx + \int_\Omega \nabla u_1\cdot\nabla g_1\int_0^\infty g(s)\left(1-e^{-s}\right)dsdx\\
&\qquad + \beta\int_\Omega \nabla g_3\cdot u_1 dx - \mu_2\tau e^{-\tau}\int_\Omega u_1\cdot\int_0^1 e^{\tau\alpha} g_5(x,\alpha)d\alpha dx\\
&\qquad - \int_\Omega \nabla u_1\cdot\int_0^\infty g(s)e^{-s}\int_0^s e^y\nabla g_6(x,y)dydsdx, \quad \forall u_1 \in (H^1_0(\Omega))^n.
\end{aligned}$$

By using (4.16) – (4.19) and the definition of $\tilde{\mu}$ and $\tilde{l}$, we have

$$\begin{aligned}
&l\int_\Omega \nabla u\cdot\nabla u_1 dx + \int_\Omega\left(\int_0^\infty g(s)\nabla w ds\right)\cdot\nabla u_1 dx\\
&\quad = -\int_\Omega\Big(-(\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\psi(1+\mu_1)\varphi + \mu_2 z(.,1) - g_2\Big)\cdot u_1, \quad \forall u_1 \in (H^1_0(\Omega))^n,
\end{aligned}$$

which is also true for all $\phi \in (C^1_0(\Omega))^n \subset (H^1_0(\Omega))^n$. Hence, we get

$$l\Delta u + \int_0^\infty g(s)\Delta w(s)ds = -(\mu+\lambda)\nabla\operatorname{div} u + \beta\nabla\psi + (1+\mu_1)\varphi + \mu_2 z(.,1) - g_2 \in \left(L^2(\Omega)\right)^n.$$

By the regularity theory for the linear elliptic equations, it follows that

$$lu + \int_0^\infty g(s)w(s)ds \ \in (H^2(\Omega))^n.$$

Similarly, if $u_1 \equiv 0 \in (H_0^1(\Omega))^n$, we obtain

$$-\kappa\Delta v - \delta\Delta\psi + \psi = g_4 - \beta \operatorname{div} \varphi \in \left(L^2(\Omega)\right)^n,$$

and by the regularity theory for the linear elliptic equations, it follows that

$$\kappa v + \delta\psi \ \in H^2(\Omega).$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $\Phi \in D(\mathcal{A})$ such that (4.14) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal, and this completes the proof. ∎

4.4 Technical Lemmas

In this section we establish several lemmas needed for the proof of our decay result.

Lemma 4.1 *Let (u, v, z, η^t) be the solution of* (4.7). *Then the energy functional, defined by* (4.9), *satisfies, for all $t \geq 0$,*

$$E'(t) \leq -m_1 \left(\int_\Omega |u_t|^2 dx + \int_\Omega z^2(x,1,t)dx \right) + \frac{1}{2}(g' o \nabla\eta^t)(t) - \delta \int_\Omega |\nabla v_t|^2 dx \leq 0, \tag{4.24}$$

for some constant m_1, where $m_1 > 0$ if $|\mu_2| < \mu_1$ and $m_1 = 0$ if $|\mu_2| = \mu_1$.

Proof. A multiplication of the first and the second equation in (4.7) by u_t and v_t, respectively, integration over Ω and using integration by parts and the boundary conditions, yield

$$\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_t|^2dx+\int_\Omega v_t^2dx+l\int_\Omega |\nabla u|^2dx+\kappa\int_\Omega |\nabla v|^2dx+(g\circ\nabla\eta^t)(t)$$
$$+(\mu+\lambda)\int_\Omega (\operatorname{div} u)^2dx\Big\}$$

$$=\frac{1}{2}(g'o\nabla\eta^t)(t)-\delta\int_\Omega |\nabla v_t|^2dx-\mu_1\int_\Omega |u_t|^2dx-\mu_2\int_\Omega u_t\cdot z(x,1,t)dx. \quad (4.25)$$

Now, multiplying the third equation in (4.7) by ξz and integrating over $\Omega\times(0,1)$, we obtain

$$\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx=-\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx+\frac{\xi}{2\tau}\int_\Omega |u_t|^2dx. \quad (4.26)$$

A combination of (4.25) – (4.26) and the use of Young's inequality, lead to

$$E'(t)\leq\frac{1}{2}(g'o\nabla\eta^t)(t)-\delta\int_\Omega |\nabla v_t|^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega |u_t|^2dx$$
$$-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega z^2(x,1,t)dx.$$

Consequently, using (4.8), estimate (4.24) follows. ∎

Lemma 4.2 *Suppose (A) holds and assume $\eta_0\in L^2_g(\mathbb{R}_+,(H^1_0(\Omega))^n)$, where*

$$L^2_g(\mathbb{R}_+,(H^1_0(\Omega))^n)=\Big\{u:\mathbb{R}_+\longrightarrow(H^1_0(\Omega))^n/\int_0^\infty g(s)\|\ \nabla\eta^t(x,s)\ \|^2_{(L^2(\Omega))^n}ds<\infty\Big\}.$$

Furthermore, we assume that there exists $N_0\geq 0$ for which

$$\int_\Omega |\nabla f_0(x,s)|^2 dx \le N_0,\ \forall s > 0. \tag{4.27}$$

Then, there exists $\beta_1 > 0$ such that for any $\delta_0 > 0$ and $t \in \mathbb{R}_+$, we have

$$G'(\delta_0 E(t)) g \circ \nabla \eta^t(t) \le -\beta_1 E'(t) + \beta_1 \delta_0 E(t) G'(\delta_0 E(t)). \tag{4.28}$$

Proof. We will use the approach of [28] to prove this lemma.

For all $t, s \in \mathbb{R}_+$, we have

$$\begin{aligned}
\int_\Omega |\nabla \eta^t(x,s)|^2 dx &= \int_\Omega |\nabla u(x,t) - \nabla u(x,t-s)|^2 dx \\
&\le 2\int_\Omega |\nabla u(x,t)|^2 dx + 2\int_\Omega |\nabla u(x,t-s)|^2 dx \\
&\le 2\int_\Omega |\nabla u(x,t)|^2 dx + 2\sup_{\tau\in\mathbb{R}} \int_\Omega |\nabla u(x,\tau)|^2 dx \\
&\le 4\int_\Omega |\nabla u(x,t)|^2 dx + 2\sup_{\tau>0} \int_\Omega |\nabla f_0(x,\tau)|^2 dx. \qquad (4.29)
\end{aligned}$$

By using (4.9) and the fact that E is nonincreasing, it follows that

$$\int_\Omega |\nabla u(x,t)|^2 dx \le \frac{2}{l} E(t) \le \frac{2}{l} E(0), \qquad \forall t \in \mathbb{R}_+ \tag{4.30}$$

By substituting (4.27) and (4.30) into (4.29), we get

$$\int_\Omega |\nabla \eta^t(x,s)|^2 dx \le \frac{8}{l} E(0) + 2N_0 \le c \tag{4.31}$$

Next, we define $K(s) = \frac{s}{G^{-1}(s)}$ for $s \in \mathbb{R}_+$, and by using (A), it follows that

$\lim_{s\to 0^+} \frac{s}{G^{-1}(s)} = \lim_{t\to 0^+} \frac{G(t)}{t} = G'(0) = 0$. Thus, $K(0) = 0$. Furthermore, by using the fact that G^{-1} is concave and $G^{-1}(0) = 0$, then for any $0 \le s_1 < s_2$, we have

$$\begin{aligned} G^{-1}(s_1) = G^{-1}\left(\frac{s_1}{s_2}s_2 + \left(1-\frac{s_1}{s_2}\right)0\right) &\ge \frac{s_1}{s_2}G^{-1}(s_2) + \left(1-\frac{s_1}{s_2}\right)G^{-1}(0) \\ &\ge \frac{s_1}{s_2}G^{-1}(s_2). \end{aligned}$$

Thus, $K(s)$ yields

$$K(s_1) \le \frac{s_1}{G^{-1}(s_1)} \le \frac{s_2}{G^{-1}(s_2)} = K(s_2).$$

Therefore, K is nondecreasing. On the other hand, let $\delta_0, \tau_1, \tau_2 > 0$, then by exploiting (4.31) and the fact that K is nondecreasing, we obtain

$$K\left(-\tau_2 g'(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \le K\left(-c\tau_2 g'(s)\right). \tag{4.32}$$

Now,

$$\begin{aligned} g\circ\nabla\eta^t(s) = &\int_0^\infty g(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dxds \\ =&\frac{1}{\tau_1 G'(\delta_0 E(t))}\int_0^\infty G^{-1}\left(-\tau_2 g'(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \\ &\times \frac{\tau_1 G'(\delta_0 E(t)) g(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dx}{G^{-1}\left(-\tau_2 g'(s)\int_\Omega |\nabla\eta^t(x,s)|^2 dx\right)} ds. \end{aligned}$$

By using the definition of K and (4.32), we get

$$
\begin{aligned}
g \circ \nabla\eta^t(s) =& \frac{1}{\tau_1 G'(\delta_0 E(t))} \int_0^\infty G^{-1}\left(-\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \\
& \times \frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) ds \\
\leq& \frac{1}{\tau_1 G'(\delta_0 E(t))} \int_0^\infty G^{-1}\left(-\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \\
& \times \frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-c\tau_2 g'(s)\right) ds.
\end{aligned}
$$

Using the definition of K again, yields

$$
\begin{aligned}
g \circ \nabla\eta^t(s) \leq& \frac{1}{\tau_1 G'(\delta_0 E(t))} \int_0^\infty G^{-1}\left(-\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \\
& \times \frac{c\tau_1 G'(\delta_0 E(t)) g(s)}{G^{-1}\left(-c\tau_2 g'(s)\right)} ds.
\end{aligned}
$$

Now, let $G^\star$ be the dual function of the convex function G defined by

$$
G^\star(t) = \sup_{s\in\mathbb{R}_+} \{ts - G(s)\} = t(G')^{-1}(t) - G((G')^{-1}(t)) \qquad \forall t \in \mathbb{R}_+,
$$

then, using the general Young's inequality $t_1 t_2 \leq G(t_1) + G^\star(t_2)$, for

$$
t_1 = G^{-1}\left(-\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx\right) \quad \text{and} \quad t_2 = \frac{c\tau_1 G'(\delta_0 E(t)) g(s)}{G^{-1}\left(-c\tau_2 g'(s)\right)},
$$

gives

$$
\begin{aligned}
g \circ \nabla\eta^t(s) \leq& \frac{1}{\tau_1 G'(\delta_0 E(t))} \Bigg(\int_0^\infty -\tau_2 g'(s) \int_\Omega |\nabla\eta^t(x,s)|^2 dx ds \\
& + \int_0^\infty G^\star\left(\frac{c\tau_1 G'(\delta_0 E(t)) g(s)}{G^{-1}\left(-c\tau_2 g'(s)\right)}\right) ds \Bigg).
\end{aligned}
$$

By exploiting (4.24) and the fact that $G^\star(s) \le s(G')^{-1}(s)$, we get

$$
\begin{aligned}
g \circ \nabla\eta^t(s) \le & \frac{-2\tau_2}{\tau_1 G'(\delta_0 E(t))} E'(t) \\
& + c \int_0^\infty \frac{g(s)}{G^{-1}\left(-c\tau_2 g'(s)\right)} (G')^{-1} \left(\frac{c\tau_1 G'(\delta_0 E(t)) g(s)}{G^{-1}\left(-c\tau_2 g'(s)\right)} \right) ds.
\end{aligned}
$$

Choosing $\tau_2 = \frac{1}{c}$ yields

$$
\begin{aligned}
g \circ \nabla\eta^t(s) \le & \frac{-2}{c\tau_1 G'(\delta_0 E(t))} E'(t) \\
& + c \int_0^\infty \frac{g(s)}{G^{-1}\left(-g'(s)\right)} (G')^{-1} \left(\frac{c\tau_1 G'(\delta_0 E(t)) g(s)}{G^{-1}\left(-g'(s)\right)} \right) ds.
\end{aligned}
$$

From (4.6), we deduce that

$$
\sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}\left(-g'(s)\right)} = \gamma_1 < \infty.
$$

In addition, using the fact that $(G')^{-1}$ is nondecreasing, we get

$$
g \circ \nabla\eta^t(s) \le \frac{-2}{c\tau_1 G'(\delta_0 E(t))} E'(t) + c(G')^{-1}\left(c\gamma_1\tau_1 G'(\delta_0 E(t))\right) \int_0^\infty \frac{g(s)}{G^{-1}\left(-g'(s)\right)} ds.
$$

Similarly, from (4.6), we deduce that

$$
\int_0^\infty \frac{g(s)}{G^{-1}\left(-g'(s)\right)} ds = \gamma_2 < \infty.
$$

Also, choosing $\tau_1 = \frac{1}{c\gamma_1}$, yields

$$g \circ \nabla\eta^t(s) \leq \frac{-2\gamma_1}{G'(\delta_0 E(t))} E'(t) + c\gamma_2\delta_0 E(t), \qquad \forall t \in \mathbb{R}_+,$$

which implies (4.28) with $\beta_1 = \max\{2\gamma_1,\ c\gamma_2\}$. ∎

Lemma 4.3 *Suppose that* (A) *holds and let* (u, v, z, η^t) *be the solution of* (4.7). *Then the functional*

$$F_1(t) := \int_\Omega u_t \cdot u dx + \frac{\mu_1}{2}\int_\Omega |u|^2 dx$$

satisfies, for some positive constant m_2*, the estimate*

$$\begin{aligned} F_1'(t) \leq& \int_\Omega |u_t|^2 dx + c\left(\int_\Omega v_t^2 dx + \int_\Omega z^2(x,1,t)dx + (g \circ \nabla\eta^t)(t)\right) \\ &- m_2\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega (\operatorname{div} u)^2 dx\right). \end{aligned} \tag{4.33}$$

Proof. Direct computations, using the first equation in (4.7), yields

$$\begin{aligned} F_1'(t) =& \int_\Omega |u_t|^2 dx - l\int_\Omega |\nabla u|^2 dx - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 dx + \beta\int_\Omega v_t \cdot \operatorname{div} u dx \\ &- \int_\Omega \nabla u \cdot \int_0^\infty g(s)\nabla\eta^t(s) ds dx - \mu_2\int_\Omega z(x,1,t)\cdot u dx. \end{aligned}$$

By using Young's and Poincaré's inequalities, we get, for $\delta_1 > 0$,

$$
\begin{aligned}
F_1'(t) \leq & \int_\Omega |u_t|^2 dx - (l - \delta_1(1+|\mu_2|)) \int_\Omega |\nabla u|^2 dx \\
& - (\mu + \lambda - \beta\delta_1) \int_\Omega (\operatorname{div} u)^2 dx + \frac{\beta}{4\delta_1} \int_\Omega v_t^2 dx \\
& + \frac{c|\mu_2|}{4\delta_1} \int_\Omega z^2(x,1,t) dx + \frac{(\mu - l)}{4\delta_1} (g \circ \nabla \eta^t)(t).
\end{aligned} \tag{4.34}
$$

By taking δ_1 small enough, (4.33) follows. ∎

Lemma 4.4 *let* (u, v, z, η^t) *be the solution of* (4.7). *Then the functional*

$$
F_2(t) := \int_\Omega v_t v dx + \beta \int_\Omega v \operatorname{div} u dx + \frac{\delta}{2} \int_\Omega |\nabla v|^2 dx
$$

satisfies, for any positive constant δ_2, *the estimate*

$$
F_2'(t) \leq -\kappa \int_\Omega |\nabla v|^2 dx + c\left(1 + \frac{1}{\delta_2}\right) \int_\Omega v_t^2 dx + c\delta_2 \int_\Omega (\operatorname{div} u)^2 dx. \tag{4.35}
$$

Proof. See the proof of Lemma 3.3 (page 45-45). ∎

As in [82], we state the following lemma.

Lemma 4.5 *let* (u, v, z, η^t) *be the solution of* (4.7). *Then the functional*

$$
F_3(t) := \tau \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x, \rho, t) d\rho dx
$$

satisfies, for some positive constant m_3, *the estimate*

$$
F_3'(t) \leq -m_3 \left(\int_\Omega z^2(x,1,t) dx + \tau \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx \right) + \int_\Omega |u_t|^2 dx. \tag{4.36}
$$

Proof. See the proof of Lemma 3.4 (page 45). ∎

Lemma 4.6 *Suppose that (A) holds and let* (u, v, z, η^t) *be the solution of* (4.7). *Then for* $|\mu_2| = \mu_1$, *the functional*

$$F_4(t) := -\int_\Omega u_t \cdot \int_0^\infty g(s)\eta^t(s)dsdx,$$

satisfies, for some positive constants m_4 *and for any* $\delta_2 > 0$, *the estimate*

$$\begin{aligned} F_4'(t) \le -m_4 \int_\Omega |u_t|^2 dx + c\delta_2 \int_\Omega |\nabla u|^2 dx + c\int_\Omega |\nabla v_t|^2 dx + c\delta_2 \int_\Omega z^2(x,1,t)dx \\ + c\delta_2 \int_\Omega (\operatorname{div} u)^2 dx + c\left(1 + \frac{1}{\delta_2}\right)(g \circ \nabla\eta^t)(t) - c(g' o \nabla\eta^t)(t). \end{aligned} \tag{4.37}$$

Proof. Differentiation of F_4, using (4.7) and integration by parts together with the boundary conditions, yield

$$\begin{aligned} F_4'(t) =& l\int_\Omega \nabla u \cdot \int_0^\infty g(s)\nabla\eta^t(s)dsdx + (\mu+\lambda)\int_\Omega \operatorname{div} u \left(\int_0^\infty g(s)\operatorname{div}\eta^t(s)ds\right)dx \\ &+ \beta\int_\Omega \nabla v_t \cdot \left(\int_0^\infty g(s)\eta^t(s)ds\right)dx + \int_\Omega \left(\int_0^\infty g(s)\nabla\eta^t(s)ds\right)^2 dx \\ &+ \mu_1 \int_\Omega u_t \cdot \int_0^\infty g(s)\eta^t(s)dsdx - \int_\Omega u_t \cdot \int_0^\infty g'(s)\eta^t(s)dsdx \\ &+ \mu_2 \int_\Omega z(x,1,t)\cdot \int_0^\infty g(s)\eta^t(s)dsdx - (\mu - l)\int_\Omega |u_t|^2 dx. \end{aligned} \tag{4.38}$$

Now, we estimate the terms in the right hand side of (4.38), exploiting Young's, Cauchy-Schwarz, and Poincaré's inequalities. So, we obtain, for $\delta_2, \delta_3 > 0$,

$$I_1 = \int_\Omega \nabla u \cdot \int_0^\infty g(s)\nabla\eta^t(s)dsdx \le \delta_2 \int_\Omega |\nabla u|^2 dx + \frac{\mu - l}{4\delta_2}(g \circ \nabla\eta^t)(t), \tag{4.39}$$

$$
\begin{aligned}
I_2 &= \int_\Omega \operatorname{div} u \left(\int_0^\infty g(s) \operatorname{div} \eta^t(s) ds\right) dx \\
&\le \delta_2 \int_\Omega (\operatorname{div} u)^2 dx + \frac{\mu - l}{4\delta_2} \int_\Omega \int_0^\infty g(s) \big(\operatorname{div} \eta^t(s)\big)^2 ds dx \qquad (4.40)\\
&\le \delta_2 \int_\Omega (\operatorname{div} u)^2 dx + \frac{\mu - l}{2\delta_2} (g \circ \nabla \eta^t)(t),
\end{aligned}
$$

$$
I_3 = \int_\Omega \nabla v_t \cdot \left(\int_0^\infty g(s) \eta^t(s) ds\right) dx \le \frac{1}{2} \int_\Omega |\nabla v_t|^2 dx + \frac{c(\mu - l)}{2} (g \circ \nabla \eta^t)(t), \quad (4.41)
$$

$$
I_4 = \int_\Omega \left(\int_0^\infty g(s) \nabla \eta^t(s) ds\right)^2 dx \le (\mu - l)(g \circ \nabla \eta^t)(t), \quad (4.42)
$$

$$
I_5 = \int_\Omega u_t \cdot \int_0^\infty g(s) \eta^t(s) ds dx \le \delta_3 \int_\Omega |u_t|^2 dx + \frac{c(\mu - l)}{4\delta_3} (g \circ \nabla \eta^t)(t), \quad (4.43)
$$

$$
\begin{aligned}
I_6 = &- \int_\Omega u_t \cdot \int_0^\infty g'(s) \eta^t(s) ds dx \le \delta_3 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_3} \int_\Omega \left(\int_0^\infty g'(s) \eta^t(s)\right)^2 dx \\
&\le \delta_3 \int_\Omega |u_t|^2 dx + \frac{1}{4\delta_3} \int_\Omega \left(\int_0^\infty -g'(s) ds\right) \left(\int_0^\infty -g'(s) |\eta^t(s)|^2 ds\right) dx \\
&\le \delta_3 \int_\Omega |u_t|^2 dx - \frac{cg(0)}{4\delta_3} (g' o \nabla \eta^t)(t)
\end{aligned} \qquad (4.44)
$$

and

$$
I_7 = \int_\Omega z(x,1,t) \cdot \int_0^\infty g(s) \eta^t(s) ds dx \le \delta_2 \int_\Omega z^2(x,1,t) dx + \frac{c(\mu - l)}{4\delta_2} (g \circ \nabla \eta^t)(t). \qquad (4.45)
$$

A combination of (4.38)−(4.45) and recalling that $|\mu_2| = \mu_1$, lead to

$$
\begin{aligned}
F_4'(t) \le &- [(\mu - l) - \delta_3(1 + \mu_1)] \int_\Omega |u_t|^2 dx + \delta_2 l \int_\Omega |\nabla u|^2 dx + \delta_2(\mu + \lambda) \int_\Omega (\operatorname{div} u)^2 dx \\
&+ \frac{\beta}{2} \int_\Omega |\nabla v_t|^2 dx + \delta_2 \mu_1 \int_\Omega z^2(x,1,t) dx - \frac{cg(0)}{4\delta_3} (g' o \nabla \eta^t)(t) \\
&+ (\mu - l) \left[1 + \frac{l}{4\delta_2} + \frac{\mu + \lambda}{2\delta_2} + \frac{c\beta}{2} + \frac{c\mu_1}{4} \left(\frac{1}{\delta_2} + \frac{1}{\delta_3}\right)\right] (g \circ \nabla \eta^t)(t).
\end{aligned}
$$

Next, we choose δ_3 small enough to get (4.37). ∎

4.5 Asymptotic Stability

This section is divided into two parts. In the first part, we discuss the case where $|\mu_2| < \mu_1$, and in the second part, we discuss the case where $|\mu_2| = \mu_1$.

4.5.1 General Decay Result for $|\mu_2| < \mu_1$

For $\varepsilon > 0$, to be chosen appropriately later, we let

$$\mathcal{L}(t) := E(t) + \varepsilon F_1(t) + \varepsilon F_2(t) + \varepsilon F_3(t). \tag{4.46}$$

Lemma 4.7 *There exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \ \forall t \geq 0, \tag{4.47}$$

for ε small enough.

Proof. See the proof of Lemma 3.6 (page 51). ∎

Theorem 4.2 *Let (u, v, z, η^t) be the solution of* (4.7)*. Assume $|\mu_2| < \mu_1$ and (A) hold. Assume further that* (4.27) *holds in case of* (4.6)*. Then, there exist two positive constants c_0 and c_1 such that the energy functional* (4.9) *satisfies*

$$E(t) \leq c_0 G_1^{-1}(c_1 t), \ \forall t \geq 0, \tag{4.48}$$

where

$$G_1(y) = \int_y^1 \frac{1}{G_0(s)} ds, \text{ on } (0,1]$$

and

$$G_0(t) = \begin{cases} t & \text{if (4.5) holds,} \\ \\ tG'(\delta_0 t) & \text{if (4.6) holds.} \end{cases}$$

Proof. By differentiating (4.46) and using (4.24), (4.33), (4.35), (4.36) and Poincaré's inequality, we obtain

$$\begin{aligned} \mathcal{L}'(t) \leq & -[m_1 - \varepsilon] \int_\Omega |u_t|^2 dx - \varepsilon m_2 \int_\Omega |\nabla u|^2 dx - \varepsilon\kappa \int_\Omega |\nabla v|^2 dx \\ & - \varepsilon [m_2 - c\delta_2] \int_\Omega (\operatorname{div} u)^2 dx - \varepsilon m_3 \tau \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx \\ & + \varepsilon c (g \circ \nabla \eta^t)(t) - \left[\delta - c\varepsilon \left(1 + \frac{1}{\delta_2}\right)\right] \int_\Omega |\nabla v_t|^2 dx \\ & - [(m_1 - \varepsilon c) + \varepsilon m_3] \int_\Omega z^2(x,1,t) dx. \end{aligned}$$

At this point, we choose δ_2 small enough so that $(m_2 - c\delta_2) > 0$,

then we choose ε so that

$$0 < \varepsilon < min \left(m_1, \ \frac{m_1}{c}, \ \frac{\delta}{c\left(1 + \frac{1}{\delta_2}\right)} \right)$$

So, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq k_1 (g \circ \nabla \eta^t)(t) - k_2 \Big\{ & \int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \\ & + \int_\Omega (\operatorname{div} u)^2 dx + \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx + \int_\Omega |\nabla v_t|^2 dx \Big\}, \end{aligned}$$

for positive constants k_1 and k_2.

Now, we use Poincaré's inequality and (4.9) to conclude that

$$\mathcal{L}'(t) \leq -k_0 E(t) + k_1 (g \circ \nabla \eta^t)(t), \ \forall t \geq 0, \tag{4.49}$$

for a positive constant k_0. Next, we estimate $(g \circ \nabla \eta^t)$ in (4.49).

For this, we discuss two cases.

Case 1. (4.5) holds. By multiplying (4.49) by γ and then using (4.5) and (4.24), we arrive at

$$(\gamma \mathcal{L}(t) + 2k_1 E(t))' \leq -k_0 \gamma E(t), \ \forall t \geq 0.$$

By exploiting (4.47), it can easily be shown that

$$\mathcal{R}(t) = \gamma \mathcal{L}(t) + 2k_1 E(t) \sim E(t). \tag{4.50}$$

Consequently, for some positive constant c_1, we obtain

$$\mathcal{R}'(t) \leq -c_1 \mathcal{R}(t), \ \forall t \geq 0. \tag{4.51}$$

A simple integration of (4.51) over $(0, t)$ leads to

$$\mathcal{R}(t) \leq \mathcal{R}(0) e^{-c_1 t}, \ \forall t \geq 0. \tag{4.52}$$

By combining (4.50) and (4.52), we obtain, for some positive constant c_0,

$$E(t) \leq c_0 e^{-c_1 t}, \ \forall t \geq 0,$$

which gives the conclusion of Theorem (4.2) since $G_1^{-1}(t) = e^{-t}$.

Case 2. (4.6) holds. As in [28], we exploit (4.28) to get

$$G'(\delta_0 E(t))\mathcal{L}'(t) \leq -k_1\beta_1 E'(t) - (k_0 - k_1\beta_1\delta_0)E(t)G'(\delta_0 E(t)), \ \forall t \geq 0,$$

By choosing δ_0 small enough so that $k_3 = (k_0 - k_1\beta_1\delta_0) > 0$, we obtain

$$G'(\delta_0 E(t))\mathcal{L}'(t) + k_1\beta_1 E'(t) \leq -k_3 E(t)G'(\delta_0 E(t)), \ \forall t \geq 0,$$

which can be rewritten as

$$(G'(\delta_0 E(t))\mathcal{L}(t) + k_1\beta_1 E(t))' - (G'(\delta_0 E(t)))'\mathcal{L}(t) \leq -k_3 E(t)G'(\delta_0 E(t)), \ \forall t \geq 0,$$

Set $\mathcal{H}(t) = G'(\delta_0 E(t))\mathcal{L}(t) + k_1\beta_1 E(t)$ and use the fact that $G'(\delta_0 E(t))$ is nonincreasing, we arrive at

$$\mathcal{H}'(t) \leq -k_3 G_0(E(t)), \ \forall t \geq 0, \tag{4.53}$$

where $G_0(t) = tG'(\delta_0 t)$. By exploiting (4.47), it can easily be shown that for some $\tilde{\alpha}_1, \tilde{\alpha}_2 > 0$, we have

$$\tilde{\alpha}_1 E(t) \le \mathcal{H}(t) \le \tilde{\alpha}_2 E(t),\ \forall t \ge 0. \tag{4.54}$$

If we let

$$\mathcal{H}_1(t) = \frac{\varepsilon_1 \mathcal{H}(t)}{\tilde{\alpha}_2},\ 0 < \varepsilon_1 < 1,$$

and taking into account (4.54), we obtain

$$\mathcal{H}_1(t) \sim E(t). \tag{4.55}$$

If we choose ε_1 small enough so that $\mathcal{H}_1(t) \le E(t)$, $\mathcal{H}_1(0) \le 1$, and note that G_0 is not decreasing, then we have, for some $c_1 > 0$,

$$\mathcal{H}_1'(t) \le -c_1 G_0(\mathcal{H}_1(t)),$$

which means

$$-\frac{\mathcal{H}_1'(t)}{G_0(\mathcal{H}_1(t))} \ge c.$$

Consequently, we get

$$(G_1(\mathcal{H}_1(t)))' \ge c_1, \tag{4.56}$$

where $G_1(t) = \int_t^1 \frac{1}{G_0(s)} ds,$ on $(0,1]$.

A simple integration of (4.56) over $(0,t)$ and using the fact that $G_1(1) = 0$, gives

$$G_1(\mathcal{H}_1(t)) \ge c_1 t,\ \forall t \ge 0.$$

Using the property (nonincreasing) of G_1, we get

$$\mathcal{H}_1(t) \le G_1^{-1}(c_1 t), \ \forall t \ge 0. \tag{4.57}$$

The conclusion of the theorem follows by combining (4.55) and (4.57). ∎

4.5.2 General Decay Result for $|\mu_2| = \mu_1$

By recalling (4.8), we have $\xi = \tau\mu_1$. Hence, (4.24) takes the form

$$E'(t) \le -\delta \int_\Omega |\nabla v_t|^2 dx + \frac{1}{2}(g' o \nabla \eta^t)(t) \le 0, \ \forall t \ge 0. \tag{4.58}$$

We then use (4.33), (4.35) and (4.36) with $|\mu_2| = \mu_1$ and define another Lyapunov functional

$$\tilde{\mathcal{L}}(t) := NE(t) + \varepsilon_2 F_1(t) + F_2(t) + \varepsilon_3 F_3(t) + F_4(t), \tag{4.59}$$

where N, ε_2 and ε_3 are positive real numbers, which will be chosen properly later.

Lemma 4.8 *For N large enough, $\tilde{\mathcal{L}}(t)$ and $E(t)$ satisfy*

$$\alpha_3 E(t) \le \tilde{\mathcal{L}}(t) \le \alpha_4 E(t), \ \forall t \ge 0, \tag{4.60}$$

for two positive constants α_3 and α_4.

Proof. The lemma is established by following the same steps enumerated in the proof of Lemma 3.6 (see page 50). ∎

Theorem 4.3 *Let (u, v, z, η^t) be the solution of* (4.7)*. Assume $|\mu_2| = \mu_1$ and* (A) *hold. Assume further that* (4.27) *holds in case of* (4.6)*. Then, there exist two positive constants $\tilde{c_0}$ and $\tilde{c_1}$ such that the energy functional* (4.9) *satisfies*

$$E(t) \leq \tilde{c_0} G_1^{-1}(t\tilde{c_1}), \ \forall t \geq 0, \tag{4.61}$$

where G_1 is defined in Theorem 4.2.

Proof. Differentiating $\tilde{\mathcal{L}}$ and using (4.33),(4.35)–(4.37), (4.58) and Poincaré's inequality, we obtain

$$\begin{aligned}
\tilde{\mathcal{L}}'(t) \leq & - [m_4 - \varepsilon_3 - \varepsilon_2] \int_\Omega |u_t|^2 dx - [\varepsilon_2 m_2 - c\delta_2] \int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega |\nabla v|^2 dx \\
& - \varepsilon_3 m_3 \tau \int_\Omega \int_0^1 z^2(x, \rho, t) d\rho dx - [\varepsilon_2 m_2 - c\delta_2] \int_\Omega (\operatorname{div} u)^2 dx \\
& - \left[N\delta - c\left(1 + \varepsilon_2 + \frac{1}{\delta_2}\right)\right] \int_\Omega |\nabla v_t|^2 dx + \left[\frac{N}{2} - c\right] (g' o \nabla \eta^t)(t) \\
& - [\varepsilon_3 m_3 - c\varepsilon_2 - c\delta_2] \int_\Omega z^2(x, 1, t) dx + c\left[1 + \varepsilon_2 + \frac{1}{\delta_2}\right] (g \circ \nabla \eta^t)(t).
\end{aligned}$$

Now, we let $\varepsilon_3 = \frac{m_4}{2}$ and then choose ε_2 small enough so that

$$k_3 = \frac{m_3 m_4}{2} - \varepsilon_2 c > 0 \ \text{ and } \ \frac{m_4}{2} - \varepsilon_2 > 0.$$

Once ε_2 fixed, we then pick δ_2 such that

$$c\delta_2 < \min(\varepsilon_2 m_2, \ k_3).$$

Finally, we choose N large enough so that (4.60) remain valid and

$$k_4 = N\delta - c\left(1 + \frac{1}{\delta_2} + \varepsilon_2\right) > 0 \text{ and } \frac{N}{2} - c > 0.$$

Hence, we arrive at

$$\mathcal{L}'(t) \leq \tilde{k}_1(g \circ \nabla\eta^t)(t) - \tilde{k}_2\left\{\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \right.$$
$$\left. + \int_\Omega (\operatorname{div} u)^2 dx + \int_\Omega \int_0^1 z^2(x,\rho,t) d\rho dx + \int_\Omega |\nabla v_t|^2 dx\right\},$$

for two positive constants $\tilde{k}_1$ and $\tilde{k}_2$.

By using Poincaré's inequality, we get, for some positive constant $\tilde{k}_0$,

$$\tilde{\mathcal{L}}'(t) \leq -\tilde{k}_0 E(t) + \tilde{k}_1(g \circ \nabla\eta^t)(t), \ \forall t \geq t_0, \tag{4.62}$$

By repeating the same steps, from (4.49) to (4.57), estimate (4.61) is obtained. ∎

Remark 4.2 Similarly to Pignotti [95], we did not require that μ_2 be positive

Remark 4.3 Our result extends the results of Pignotti [95], and Said-Houari and Rahali [105]. In fact, in [105], the authors considered only relaxation functions of exponential decay type and required that $\mu_2 > 0$, while in our case, μ_2 is a real number satisfying $|\mu_2| \leq \mu_1$ and the relaxation function is of a general decay.

We now give an example to illustrate our general decay estimate.

Let $g(t) = ae^{-bt}$ and $G(t) = t^{p+1}$. Assumption (A) holds for $a < \mu b$ and $p > 0$.

For (4.5), we have $\gamma = b$ and (4.48) implies

$$E(t) \leq c_0 e^{-c_1 t}, \ \forall t \geq 0,$$

whereas in case of (4.6), we have

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p}}}, \ \forall t \geq 0,$$

for some $C > 0$. We refer the reader to [28] and [31] for more examples.

CHAPTER 5

ONE-DIMENSIONAL SYSTEM OF THERMOELASTICITY OF TYPE III WITH A DELAY TERM

In this chapter, we consider a one-dimensional system of thermoelasticity of type III with a constant internal delay. That is

$$
\begin{cases}
u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta\theta_x(x,t) + \mu u_t(x,t-\tau) = 0, & x \in (0,1),\ t > 0, \\
\theta_{tt}(x,t) - \kappa\theta_{xx}(x,t) - \delta\theta_{xxt}(x,t) + \beta u_{xtt}(x,t), = 0, & x \in (0,1),\ t > 0, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in (0,1), \\
u_t(x,-t) = f_0(x,t), & x \in (0,1),\ t \in (0,\tau), \\
u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, & t \geq 0,
\end{cases}
\tag{5.1}
$$

where $u(x,t)$ is the displacement, $\theta(x,t)$ is the difference temperature, the coefficients $\alpha,\ \beta,\ \kappa,\ \delta$ are positive constants, μ is a real number, $\tau > 0$ represents the time delay, $u_0, u_1, \theta_0, \theta_1$ are initial data and f_0 is a history function. The well-posedness of the problem is consider in section 5.1 and the exponential decay of the energy is establish in section 5.2.

5.1 The Well-posedness of the Problem

In this section, we give the existence and uniqueness result for problem (5.1) using the semi-group theory. To this end, we first transform (5.1) into an equivalent problem by introducing some new dependent variables as in [117]:

$$
v(x,t) = \int_0^t \theta(x,s)ds + \chi(x), \tag{5.2}
$$

where $\chi \in H^1(0,1)$ is the solution of

$$
\begin{cases}
-\kappa\chi'' = \delta\theta_0'' - \theta_1 - \beta u_1' \text{ in } (0,1), \\
\\
\chi'(0) = \chi'(1) = 0.
\end{cases}
\tag{5.3}
$$

Then, integrating the second equation in (5.1) with respect to t and using (5.2) and (5.3), we get

$$v_{tt} - \kappa v_{xx} - \delta v_{xxt} + \beta u_{xt} = 0.$$

Next, we introduce, as in [82], another new dependent variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \text{ in } (0,1) \times (0,1) \times (0,\infty).$$

A simple differentiation shows that z satisfies

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \qquad x \in (0,1),\ \rho \in (0,1),\ t > 0.$$

Hence, problem (5.1) takes the form

$$
\begin{cases}
u_{tt}(x,t) - \alpha u_{xx}(x,t) + \beta v_{xt}(x,t) + \mu z(x,1,t) = 0, & x \in (0,1),\ t > 0, \\
v_{tt}(x,t) - \kappa v_{xx}(x,t) - \delta v_{xxt}(x,t) + \beta u_{xt}(x,t) = 0, & x \in (0,1),\ t > 0, \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in (0,1),\ \rho \in (0,1), t > 0, \\
z(x,0,t) = u_t(x,t), & x \in (0,1),\ t > 0, \\
u(x,0) = u_0(x), u_t(x,0) = u_1(x), v(x,0) = v_0(x), v_t(x,0) = v_1(x), & x \in (0,1), \\
z(x,\rho,0) = f_0(x,\tau\rho), & x \in (0,1),\ \rho \in (0,1), \\
u(0,t) = u(1,t) = v_x(0,t) = v_x(1,t) = 0, & t \geq 0.
\end{cases}
\tag{5.4}
$$

Thus, we shall consider problem (5.4) instead of (5.1). In order to be able to use Poincaré's inequality for v, let

$$
\hat{v}(x,t) = v(x,t) - t\int_0^1 v_1(x)dx - \int_0^1 v_0(x)dx,
$$

hence, using the second equation in (5.4), we obtain

$$
\int_0^1 \hat{v}(x,t)dx = 0, \quad \forall t \geq 0. \tag{5.5}
$$

Therefore, Poincaré's inequality is applicable for $\hat{v}$ and, in addition, $(u,\hat{v},z)$ satisfies (5.4). From now on, we work with $\hat{v}$ but write v for convenience.

We set

$$L^2_\star(0,1) = \left\{ w \in L^2(0,1) \ \mid \ \int_0^1 w(s)ds = 0 \right\},$$

$$H^1_\star(0,1) = H^1(0,1) \cap L^2_\star(0,1)$$

$$H^2_\star(0,1) = \{w \in H^2(0,1) : w_x(0) = w_x(1) = 0\}$$

and introduce the Hilbert space

$$\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1) \times L^2\big((0,1), L^2(0,1)\big).$$

equipped with the inner product

$$(\Phi, \tilde{\Phi})_{\mathcal{H}} = \alpha \int_0^1 u_x \tilde{u}_x dx + \int_0^1 \varphi \tilde{\varphi} dx + \kappa \int_0^1 v_x \tilde{v}_x dx + \int_0^1 \psi \tilde{\psi} dx + \tau |\mu| \int_0^1 \int_0^1 z \tilde{z} d\rho dx, \tag{5.6}$$

for $\Phi = (u, \varphi, v, \psi, z)^T$ and $\tilde{\Phi} = (\tilde{u}, \tilde{\varphi}, \tilde{v}, \tilde{\psi}, \tilde{z})^T \in \mathcal{H}$. It is easy to check that $\mathcal{H}$, with respect to (5.6), forms a Hilbert space.

Now, with $\Phi = (u, \varphi, v, \psi, z)^T$, where $\varphi = u_t,$ and $\psi = v_t$, system (5.4) can be rewritten in the following form:

$$\begin{cases} \Phi'(t) + (\mathcal{A} + \mathcal{B})\Phi(t) = 0, \ t > 0, \\ \Phi(0) = \Phi_0 = (u_0, u_1, v_0, v_1, f_0), \end{cases} \tag{5.7}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -\varphi \\ \\ -\alpha u_{xx} + \beta\psi_x + |\mu|\varphi + \mu z(.,1) \\ \\ -\psi \\ \\ -\kappa v_{xx} - \delta\psi_{xx} + \beta\varphi_x \\ \\ \frac{1}{\tau} z_\rho \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \begin{Bmatrix} \Phi \in \mathcal{H} \mid u \in H^2(0,1) \cap H_0^1(0,1), \quad \varphi \in H_0^1(0,1), \quad v, \psi \in H_\star^1(0,1) \\ \\ \kappa v + \delta\psi \in H_\star^2(0,1), \quad z, z_\rho \in L^2\left((0,1), L^2(0,1)\right), \quad z(x,0) = \varphi(x) \end{Bmatrix}$$

and the operator $\mathcal{B} : D(\mathcal{B}) = \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{B}\Phi = |\mu| \begin{pmatrix} 0 \\ \\ -\varphi \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{pmatrix}.$$

We have the following existence and uniqueness result:

Theorem 5.1 *Let $\Phi_0 \in \mathcal{H}$, then there exists a unique solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem* (5.7)*. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. We use the semi-group approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator and that $\mathcal{B}$ is a Lipschitz continuous operator. First, we prove that $\mathcal{A}$ is monotone. So, for any $\Phi \in D(\mathcal{A})$, we have

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = |\mu| \int_0^1 \varphi^2 dx + \delta \int_0^1 \psi_x^2 dx + \mu \int_0^1 \varphi z(.,1) dx + |\mu| \int_0^1 \int_0^1 z z_\rho d\rho dx. \quad (5.8)$$

Using Young's inequality, the third term in the right-hand side of (5.8) gives

$$-\mu \int_0^1 \varphi z(.,1) dx \leq \frac{|\mu|}{2} \int_0^1 z^2(.,1) dx + \frac{|\mu|}{2} \int_0^1 \varphi^2 dx.$$

Also, using integration by parts and the fact that $z(x,0) = \varphi(x)$, the last term in the right-hand side of (5.8) gives

$$\int_0^1 \int_0^1 z z_\rho d\rho dx = \frac{1}{2} \int_0^1 z^2(.,1) dx - \frac{1}{2} \int_0^1 \varphi^2 dx.$$

Consequently, (5.8) yields

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \geq \delta \int_0^1 {\psi_x}^2 dx.$$

Hence $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective. Given $\mathcal{G} = (g_1, g_2, g_3, g_4, g_5)^T \in \mathcal{H}$, we prove that there exists $\Phi \in D(\mathcal{A})$ satisfying

$$(I + \mathcal{A})\, \Phi = \mathcal{G}. \quad (5.9)$$

That is,

$$\begin{cases} -\varphi + u = g_1 \\ -\alpha u_{xx} + \beta\psi_x + (1+|\mu|)\varphi + \mu z(.,1) = g_2 \\ -\psi + v = g_3 \\ -\kappa v_{xx} - \delta\psi_{xx} + \beta\varphi_x + \psi = g_4 \\ z_\rho + \tau z = \tau g_5. \end{cases} \tag{5.10}$$

Suppose u and v are given with the appropriate regularity. Then, the first and the third equations in (5.10) yield

$$\varphi = u - g_1 \in H_0^1(0,1) \tag{5.11}$$

and

$$\psi = v - g_3 \in H_\star^1(0,1), \tag{5.12}$$

respectively.

The fifth equation in (5.10) together with (5.11) and the fact that $z(x,0) = \varphi(x)$ gives

$$z(x,\rho) = u(x)e^{-\tau\rho} - e^{-\tau\rho}g_1 + \tau e^{-\tau\rho}\int_0^\rho e^{\tau\alpha}g_5(x,\alpha)d\alpha. \tag{5.13}$$

Using integration by parts, it can easily be shown that the second and fourth equations in (5.10) satisfy the following:

$$
\begin{cases}
\alpha \displaystyle\int_0^1 u_x u_{1x} dx + \beta \int_0^1 \psi_x u_1 dx + (1+|\mu|) \int_0^1 \varphi u_1 dx + \mu \int_0^1 z(.,1) u_1 dx \\
\qquad\qquad\qquad\qquad = \displaystyle\int_0^1 g_2 u_1 dx, \qquad \forall u_1 \in H_0^1(0,1) \\
\kappa \displaystyle\int_0^1 v_x v_{1x} dx + \delta \int_0^1 \psi_x v_{1x} dx + \beta \int_0^1 \varphi_x v_1 dx + \int_0^1 \psi v_1 dx \\
\qquad\qquad\qquad\qquad = \displaystyle\int_0^1 g_4 v_1 dx, \qquad \forall v_1 \in H_\star^1(0,1).
\end{cases}
\tag{5.14}
$$

Furthermore, by using (5.11) – (5.13), we have the following corresponding weak formulation for the second and fourth equations in (5.10):

Finding $(u,v) \in (H_0^1(0,1))^2$ such that for all $(u_1, v_1) \in (H_0^1(0,1))^2$ the following holds:

$$
B\big((u,v),(u_1,v_1)\big) = F\big(u_1,v_1\big), \tag{5.15}
$$

where $B : \big[H_0^1(0,1) \times H_\star^1(0,1)\big]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$
\begin{aligned}
B\big((u,v),(u_1,v_1)\big) = \tilde{\mu} \int_0^1 u u_1 dx + \alpha \int_0^1 u_x u_{1x} dx + \beta \int_0^1 v_x u_1 dx + \int_0^1 v v_1 dx \\
+ (\kappa + \delta) \int_0^1 v_x v_{1x} dx + \beta \int_0^1 u_x v_1 dx,
\end{aligned}
$$

$F : H_0^1(0,1) \times H_\star^1(0,1) \longrightarrow \mathbb{R}$ is the linear functional given by

$$
\begin{aligned}
F\big(u_1,v_1\big) = \tilde{\mu} \int_0^1 g_1 u_1 dx + \int_0^1 g_2 u_1 dx + \beta \int_0^1 g_{3x} u_1 dx + \int_0^1 g_3 v_1 dx + \int_0^1 g_4 v_1 dx \\
+ \beta \int_0^1 g_{1x} v_1 dx + \delta \int_0^1 g_{3x} v_{1x} dx - \mu \tau e^{-\tau} \int_0^1 u_1 \int_0^1 e^{\tau\alpha} g_5(x,\alpha) d\alpha dx,
\end{aligned}
$$

and $\tilde{\mu} = 1 + |\mu| + \mu e^{-\tau}$.

Now, we equip $H_0^1(0,1) \times H_\star^1(0,1)$ with the following norm:

$$\|(u,v)\|^2_{H_0^1(0,1)\times H_\star^1(0,1)} = \|u\|_2^2 + \|u_x\|_2^2 + \|v\|_2^2 + \|v_x\|_2^2.$$

Similar to the steps on page 72-75, we can easily show that B and F are bounded. Furthermore, we have,

$$\begin{aligned} B\big((u,v),(u,v)\big) &= \tilde{\mu}\int_0^1 u^2dx + \alpha\int_0^1 u_x^2dx + \int_0^1 v^2dx + (\kappa+\delta)\int_0^1 v_x^2dx \\ &\geq c\|(u,v)\|^2_{H_0^1(0,1)\times H_\star^1(0,1)}, \end{aligned}$$

which means that B is coercive. Consequently, by Lax-Milgram Lemma we conclude that there exists a unique $(u,v) \in H_0^1(0,1) \times H_\star^1(0,1)$ which satisfies (5.15). Now, by substituting u into (5.11) and (5.13), and v into (5.12), we obtain

$$\varphi \in H_0^1(0,1),\ \ \psi \in H_\star^1(0,1) \text{ and } z \in L^2\left((0,1), L^2(0,1)\right).$$

Furthermore, if $(u_1,\ v_1) \equiv (u_1,\ 0) \in H_0^1(0,1) \times H_\star^1(0,1)$, then (5.15) reduces to

$$\begin{aligned} \tilde{\mu}\int_0^1 uu_1dx + \alpha\int_0^1 u_xu_{1x}dx + \beta\int_0^1 v_xu_1dx &= \tilde{\mu}\int_0^1 g_1u_1dx + \int_0^1 g_2u_1dx \\ &+ \beta\int_0^1 g_{3x}u_1dx - \mu\tau e^{-\tau}\int_0^1 u_1\int_0^1 e^{\tau\alpha}g_5(x,\alpha)d\alpha dx, \quad \forall u_1 \in H_0^1(0,1). \end{aligned}$$

By using (5.11) – (5.13), we have

$$\alpha\int_0^1 u_xu_{1x}dx + \int_0^1 \left(\beta\psi_x + (1+|\mu|)\varphi + \mu z(.,1)\right)u_1dx = \int_0^1 g_2u_1dx, \quad \forall u_1 \in H_0^1(0,1). \tag{5.16}$$

Equation (5.16) is also true for any $\phi_1 \in C_0^1(0,1) \subset H_0^1(0,1)$. Hence, we have

$$\alpha \int_0^1 u_x \phi_{1x} dx = -\int_0^1 \left(\beta\psi_x + (1+|\mu|)\varphi + \mu z(.,1) - g_2\right) \phi_1 dx, \quad \forall \phi_1 \in H_0^1(0,1).$$

Thus,

$$\alpha u_{xx} = \beta\psi_x + (1+|\mu|)\varphi + \mu z(.,1) - g_2 \in L^2(0,1),$$

which solves the second equation in (5.10). Consequently, by the elliptic regularity theory, it follows that

$$u \in H^2(0,1) \cap H_0^1(0,1).$$

Similarly, if $(u_1,\ v_1) \equiv (0,\ v_1) \in H_0^1(0,1) \times H_\star^1(0,1)$, then we obtain

$$\int_0^1 (\kappa v_x + \delta\psi_x) v_{1x} dx + \int_0^1 \left(\beta\varphi_x + \psi - g_4\right) v_1 dx = 0, \quad \forall v_1 \in H_\star^1(0,1), \tag{5.17}$$

which gives

$$\kappa v_{xx} + \delta\psi_{xx} = \beta\varphi_x + \psi - g_4 \in L^2(0,1),$$

thereby satisfying the fourth equation in (5.10). Thus, by the elliptic regularity theory, we deduce that

$$\kappa v + \delta\psi \in H^2(0,1).$$

Furthermore, (5.17) is also true for any $\phi_2 \in C^1([0,1]) \subset H_\star^1(0,1)$. Hence, we have

$$\int_0^1 (\kappa v_x + \delta\psi_x)\phi_{2x} dx + \int_0^1 \left(\beta\varphi_x + \psi - g_4\right) \phi_2 dx = 0, \quad \forall \phi_2 \in C^1([0,1]).$$

By using integration by parts, we obtain

$$(\kappa v_x(1) + \delta\psi_x(1))\,\phi_2(1) - (\kappa v_x(0) + \delta\psi_x(0))\,\phi_2(0) = 0, \quad \forall \phi_2 \in C^1([0,1]).$$

Hence,

$$\kappa v_x(1) + \delta\psi_x(1) = \kappa v_x(0) + \delta\psi_x(0) = 0.$$

Therefore,

$$\kappa v + \delta\psi \in H^2_\star(0,1).$$

Last, it is clear from (5.13) that

$$z_\rho \in L^2\left((0,1)\times(0,1)\right) \text{ and } z(x,0) = \varphi(x).$$

Hence, there exists a unique $\Phi \in D(\mathcal{A})$ such that (5.9) is satisfied. Therefore, operator $\mathcal{A}$ is maximal. With this, we conclude that $\mathcal{A}$ is a maximal monotone operator. On the other hand, from the definition of $\mathcal{B}$, it is clear (since $\mathcal{B}$ is linear) that for $\Phi_1 \in \mathcal{H}$, we have

$$|\mathcal{B}\Phi_1|^2 = |\mu|^2 \left(\int_0^1 |\varphi_1|dx\right)^2 \leq |\mu|^2 \int_0^1 |\varphi_1|^2 dx,$$

then

$$\|\mathcal{B}\Phi_1\|_{\mathcal{H}} \leq |\mu|\|\varphi_1\|_{\mathcal{H}} = |\mu|\|\Phi_1\|_{\mathcal{H}},$$

which implies that the operator $\mathcal{B}$ is Lipschitz continuous.

Consequently, $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a linear contraction C_0- semigroup on $\mathcal{H}$. Hence, the result of Theorem 5.1 follows (see [46] and [93]). ∎

5.2 Exponential Decay Result

In this section, we discuss the asymptotic behavior of the solutions of system (5.4). For the solution of problem (5.4), we define the energy functional

$$\begin{aligned} E(t) = E(t,u,v,z) =& \frac{1}{2}\int_0^1 u_t^2 dx + \frac{\alpha}{2}\int_0^1 u_x^2 dx + \frac{1}{2}\int_0^1 v_t^2 dx + \frac{\kappa}{2}\int_0^1 v_x^2 dx \\ &+ \frac{\tau|\mu|}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx. \end{aligned} \tag{5.18}$$

The main result in this section is the following:

Theorem 5.2 *Let (u,v,z) be the solution of (5.4). There exists a positive constant k_0 such that, for $|\mu|<k_0$, the energy functional (5.18) satisfies*

$$E(t) \le k_1 e^{-k_2 t},\ \forall t \ge 0, \tag{5.19}$$

for two positive constants k_1 and k_2.

In order to prove Theorem 5.2, we need several lemmas.

Lemma 5.1 *Let (u,v,z) be the solution of (5.4). Then the energy functional, defined by (5.18), satisfies*

$$E'(t) \le |\mu|\int_0^1 u_t^2 dx - \delta\int_0^1 v_{xt}^2 dx, \quad \forall t \ge 0. \tag{5.20}$$

Proof. A multiplication of the first and the second equation in (5.4) by u_t and v_t, respectively, integration over $(0,1)$ and using integration by parts and the boundary conditions, yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\int_0^1 u_t^2dx + \alpha\int_0^1 u_x^2dx + \int_0^1 v_t^2dx + \kappa\int_0^1 v_x^2dx\right\} \\ = -\delta\int_0^1 v_{xt}^2dx - \mu\int_0^1 u_t z(x,1,t)dx \\ \le -\delta\int_0^1 v_{xt}^2dx + \frac{\mu}{2}\int_0^1 u_t^2dx + \frac{\mu}{2}\int_0^1 z^2(x,1,t)dx. \end{aligned} \tag{5.21}$$

Now, multiplying the third equation in (5.4) by $|\mu|z$ and integrating over $(0,1)\times(0,1)$, we obtain

$$\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx = \frac{|\mu|}{2}\int_0^1 u_t^2dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1,t)dx. \tag{5.22}$$

Estimate (5.20) easily follows by the combination of (5.21) and (5.22). ∎

Remark 5.1 It is obvious from (5.20) that the energy E is not decreasing in general. Thus, system (5.4) is not necessarily dissipative.

Lemma 5.2 *Let (u,v,z) be the solution of (5.4). Then the functional*

$$F_1(t) := \int_0^1 u_t u dx$$

satisfies the estimate

$$F_1'(t) \le -\frac{\alpha}{2}\int_0^1 u_x^2dx + \int_0^1 u_t^2dx + c\left(\int_0^1 v_{xt}^2dx + \mu^2\int_0^1 z^2(x,1,t)dx\right). \tag{5.23}$$

Proof. Direct computations, using the first equation in (5.4), yields

$$F_1'(t) = \int_0^1 u_t^2 dx - \alpha \int_0^1 u_x^2 dx + \beta \int_0^1 u_x v_t dx - \mu \int_0^1 u z(x,1,t) dx.$$

By using Young's and Poincaré's inequalities, we get, for $\delta_1 > 0$,

$$F_1'(t) \leq \int_0^1 u_t^2 dx - \big(\alpha - \delta_1\big) \int_0^1 u_x^2 dx + \frac{\beta^2}{2\delta_1} \int_0^1 v_{xt}^2 dx + \frac{\mu^2}{2\delta_1} \int_0^1 z^2(x,1,t) dx.$$

By taking $\delta_1 = \dfrac{\alpha}{2}$, we get (5.23). ∎

Lemma 5.3 *Let (u,v,z) be the solution of (5.4). Then the functional*

$$F_2(t) := \int_0^1 v_t v dx + \beta \int_0^1 u_x v dx + \frac{\delta}{2} \int_0^1 v_x^2 dx$$

satisfies, for any positive constant δ_2, the estimate

$$F_2'(t) \leq -\kappa \int_0^1 v_x^2 dx + c\left(1 + \frac{1}{\delta_2}\right) \int_0^1 v_{xt}^2 dx + \delta_2 \int_0^1 u_x^2 dx. \tag{5.24}$$

Proof. Taking the derivative of F_2 and using the second equation in (5.4), it easily follows that

$$F_2'(t) = -\kappa \int_0^1 v_x^2 dx + \int_0^1 v_t^2 dx + \beta \int_0^1 v_t u_x dx,$$

where we have used integration by parts and the boundary conditions in (5.4). By exploiting Poincaré's and Young's inequalities for any $\delta_2 > 0$, (5.24) is established. ∎

Lemma 5.4 *Let (u, v, z) be the solution of (5.4). Then the functional (first defined in [82])*

$$F_3(t) := \tau \int_0^1 \int_0^1 e^{-\tau\rho} z^2(x, \rho, t) d\rho dx,$$

satisfies, for some positive constant m_1, the estimate

$$F_3' \leq -m_1 \left(\int_0^1 z^2(x, 1, t) dx + \tau \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx \right) + \int_0^1 u_t^2 dx. \tag{5.25}$$

Proof. See the proof of Lemma 3.4 (page 45). ∎

Lemma 5.5 *Let (u, v, z) be the solution of (5.4). Then the functional*

$$F_4(t) := \int_0^1 u_t \left(\int_0^x v_t(s, t) ds \right) dx - \kappa \int_0^1 v_x u dx$$

satisfies, for any $\delta_2 > 0$, the estimate

$$F_4'(t) \leq -\frac{\beta}{2} \int_0^1 u_t^2 dx + c\delta_2 \int_0^1 u_x^2 dx + c \left(1 + \frac{1}{\delta_2} \right) \int_0^1 v_{xt}^2 dx + \mu^2 \delta_2 \int_0^1 z^2(x, 1, t) dx. \tag{5.26}$$

Proof. Differentiation of $F_4(t)$, then using the first and second equations in (5.4), give

$$
\begin{aligned}
F_4'(t) =& \int_0^1 \Big(\alpha u_{xx} - \beta v_{xt} - \mu z(x,1,t)\Big)\Big(\int_0^x v_t(s,t)ds\Big)dx - \kappa\int_0^1 uv_{xt}dx \\
&+ \int_0^1 u_t\Big(\int_0^x (\kappa v_{xx} + \delta v_{xxt} - \beta u_{xt})ds\Big)dx - \kappa\int_0^1 u_t v_x dx.
\end{aligned}
\tag{5.27}
$$

Use of integration by parts, (5.5) and the boundary conditions in (5.4) yield

$$
\begin{aligned}
F_4'(t) =& -\beta\int_0^1 u_t^2 dx + \beta\int_0^1 v_t^2 dx + \delta\int_0^1 u_t v_{xt} dx - \kappa\int_0^1 uv_{xt}dx \\
&- \alpha\int_0^1 u_x v_t dx - \mu\int_0^1 z(x,1,t)\Big(\int_0^x v_t(s,t)ds\Big)dx.
\end{aligned}
\tag{5.28}
$$

Now, we estimate the terms in the right-hand side of (5.28), exploiting Young's, and Poincaré's inequalities. So, we obtain for $\delta_2, \delta_3 > 0$,

$$
\int_0^1 u_t v_{xt} dx \leq \delta_3\int_0^1 u_t^2 dx + \frac{c}{\delta_3}\int_0^1 v_{xt}^2 dx, \tag{5.29}
$$

$$
-\int_0^1 uv_{xt} dx \leq \delta_2\int_0^1 u_x^2 dx + \frac{c}{\delta_2}\int_0^1 v_{xt}^2 dx, \tag{5.30}
$$

$$
-\int_0^1 u_x v_t dx \leq \delta_2\int_0^1 u_x^2 dx + \frac{c}{\delta_2}\int_0^1 v_{xt}^2 dx \tag{5.31}
$$

and

$$
\begin{aligned}
-\mu\int_0^1 z(x,1,t)\Big(\int_0^x v_t(s,t)ds\Big)dx \leq& \mu^2\delta_2\int_0^1 z^2(x,1,t)dx \\
&+ \frac{c}{\delta_2}\int_0^1\Big(\int_0^x v_t(s,t)ds\Big)^2 dx.
\end{aligned}
\tag{5.32}
$$

Using Cauchy-Schwarz and Poincaré's inequalities on the last term in (5.32), gives

$$
\int_0^1\Big(\int_0^x v_t(s,t)ds\Big)^2 dx \leq \Big(\int_0^1 v_t dx\Big)^2 \leq \int_0^1 v_{xt}^2 dx.
$$

Thus, (5.32) yields

$$-\mu \int_0^1 z(x,1,t)\left(\int_0^x v_t(s,t)ds \right)dx \leq \mu^2\delta_2 \int_0^1 z^2(x,1,t)dx + \frac{c}{\delta_2}\int_0^1 v_{xt}^2 dx. \quad (5.33)$$

A combination of (5.28)−(5.31) and (5.33), leads to

$$F_4'(t) \leq -(\beta - \delta\delta_3)\int_0^1 u_t^2 dx + c\delta_2 \int_0^1 u_x^2 dx + \mu^2\delta_2 \int_0^1 z^2(x,1,t)dx$$
$$+ c\left(1 + \frac{1}{\delta_3} + \frac{1}{\delta_2}\right)\int_0^1 v_{xt}^2 dx.$$

Next, we set $\delta_3 = \dfrac{\beta}{2\delta}$ to obtain (5.26). ∎

Now, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 5.6 *The functional defined by*

$$\mathcal{L}(t) := E(t) + \varepsilon_1\left(F_1(t) + F_2(t) + F_3(t) + \frac{6}{\beta}F_4(t)\right), \quad (5.34)$$

where ε_1 is a positive real number to be chosen appropriately later, satisfies

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t),\ \forall t \geq 0, \quad (5.35)$$

for two positive constants α_1 and α_2.

Proof. See the proof of Lemma 3.6 (page 51). ∎

Proof of Theorem 5.2.

To finalize the proof of Theorem 5.2, we differentiate (5.34), and recall (5.20), (5.23)–(5.26) to obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \Big[\varepsilon_1 - |\mu|\Big] \int_0^1 u_t^2 dx - \varepsilon_1 m_1 \tau \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx \\
& - \varepsilon_1 \left[\frac{\alpha}{2} - c\delta_2\right] \int_0^1 u_x^2 dx - \kappa \varepsilon_1 \int_0^1 v_x^2 dx - \left[\delta - c\varepsilon_1 \left(1 + \frac{1}{\delta_2}\right)\right] \int_0^1 v_{xt}^2 dx \\
& - \varepsilon_1 \left[m_1 - c\mu^2 \left(1 + \delta_2\right)\right] \int_0^1 z^2(x,1,t) dx.
\end{aligned}
$$

Now, we need to carefully choose our constants. We set $\delta_2 = \frac{\alpha}{4c}$. This choice yields

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \Big[\varepsilon_1 - |\mu|\Big] \int_0^1 u_t^2 dx - \varepsilon_1 m_1 \tau \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx \\
& - \frac{\alpha}{4} \varepsilon_1 \int_0^1 u_x^2 dx - \kappa \varepsilon_1 \int_0^1 v_x^2 dx - \Big[\delta - c\varepsilon_1\Big] \int_0^1 v_{xt}^2 dx \\
& - \varepsilon_1 \Big[m_1 - c\mu^2\Big] \int_0^1 z^2(x,1,t) dx.
\end{aligned}
$$

At this point, we choose ε_1 small enough so that (5.35) remains valid and

$$
\delta - c\varepsilon_1 > 0.
$$

Finally, it is clear, for any

$$
|\mu| < k_0 = \min\left(\varepsilon_1, \sqrt{\frac{m_1}{c}}\right),
$$

we have

$$\mathcal{L}'(t) \leq -k_3 E(t), \ \forall t > 0, \tag{5.36}$$

where k_3 is a positive constant. A combination of (5.35) and (5.36) gives

$$\mathcal{L}'(t) \leq -k_2 \mathcal{L}(t), \ \forall t > 0, \tag{5.37}$$

where $k_2 = \dfrac{k_3}{\alpha_2}$. A simple integration of (5.37) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_2 t}, \ \forall t > 0. \tag{5.38}$$

Thus, using (5.35) and (5.38), the conclusion of theorem 5.1 follows.

CHAPTER 6

MEMORY-TYPE POROUS THERMOELASTIC SYSTEM OF TYPE III

In this chapter, we consider the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \theta_x = 0, & x \in (0,1),\ t > 0, \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + K\left(\varphi_x + \psi\right) - \theta + \displaystyle\int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & x \in (0,1),\ t > 0, \\ \rho_3\theta_{tt} - \kappa\theta_{xx} - \delta\theta_{xxt} + \beta\varphi_{xtt} + \beta\psi_{tt} = 0, & x \in (0,1),\ t > 0, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), & x \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0, & t \geq 0, \end{cases} \tag{6.1}$$

where $\varphi(x,t)$ is the longitudinal displacement, $\psi(x,t)$ is the volume fraction, $\theta(x,t)$ is the difference temperature, the relaxation function g is positive and decreasing, the coefficients $\rho_1, \rho_2, \rho_3, K, \alpha, \kappa, \delta$ and β are positive constants, and $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0$ and θ_1 are initial data. This is a one-dimensional porous-thermoelastic system of type III with a viscoelastic damping acting on one of the equations. We investigate system (6.1) and establish a general decay result for the case of equal as well as different speeds of wave propagation. In section 6.1, we introduce some transformations and assumptions needed in this chapter. We state and prove some technical lemmas in section 6.2. The statements with proof for the case of equal and different speeds of wave propagation will be given in section 6.3.

6.1 Assumptions and Transformations

In this section, we present some material needed in the proof of our results. For the relaxation function g, we assume the following:

$(A1)$ $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \qquad \alpha - \int_0^\infty g(s)ds = l > 0.$$

$(A2)$ There exists a positive nonincreasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\eta(t)g(t), \qquad t \geq 0.$$

Remark 6.1 There are many functions satisfying (A1) and (A2). See Remark 3.1 (page 37) for some examples.

In order to exhibit the dissipative nature of system (6.1), we introduce the new variable

$$u(x,t) = \int_0^t \theta(x,s)ds + \chi(x), \tag{6.2}$$

where $\chi(x)$ is the solution of

$$\begin{cases} -\kappa\chi'' = \delta\theta_0'' - \rho_3\theta_1 - \beta\varphi_1' - \beta\psi_1 & \text{in } (0,1), \\ \chi(0) = \chi(1) = 0. \end{cases} \tag{6.3}$$

A simple integration of the third equation in (6.1) with respect to t and taking into account (6.2) and (6.3) transform (6.1) into

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + u_{tx} = 0, & x \in (0,1),\ t > 0, \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + K(\varphi_x + \psi) - u_t + \displaystyle\int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & x \in (0,1),\ t > 0, \\ \rho_3 u_{tt} - \kappa u_{xx} - \delta u_{txx} + \beta\varphi_{tx} + \beta\psi_t = 0, & x \in (0,1),\ t > 0, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,1), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = u(0,t) = u(1,t) = 0, & t \geq 0. \end{cases} \tag{6.4}$$

The first-order energy associated with problem (6.4) is given as

$$
\begin{aligned}
E(t) = E_1(\varphi, \psi, u) = \frac{\beta}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K\left(\varphi_x + \psi\right)^2\right] dx + \frac{\beta}{2} g \circ \psi_x \\
+ \frac{1}{2}\int_0^1 \left[\rho_3 u_t^2 + \kappa u_x^2\right] dx + \frac{\beta}{2}\left(\alpha - \int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx,
\end{aligned}
\tag{6.5}
$$

where

$$
(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)\left(v(x,t) - v(x,s)\right)^2 dsdx,\ \forall v \in L^2(0,1).
$$

6.2 Technical Lemmas

In this section, we establish several lemmas needed to prove our main result.

Lemma 6.1 *Let (φ, ψ, u) be the solution of* (6.4). *Then the energy functional E, defined by* (6.5), *satisfies*

$$
E'(t) = -\delta\int_0^1 u_{xt}^2 dx + \frac{\beta}{2} g' \circ \psi_x - \frac{\beta}{2} g(t)\int_0^1 \psi_x^2 dx \leq 0. \tag{6.6}
$$

Proof. Multiplying the first equation of (6.4) by $\beta\varphi_t$, the second by $\beta\psi_t$, and the third by u_t, integrating over $(0,1)$, using integration by parts and the boundary conditions, then summing up, we obtain

$$
\begin{aligned}
\frac{\beta}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K\left(\varphi_x + \psi\right)^2 + \psi_x^2\right] dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_3 u_t^2 + \kappa u_x^2\right] dx \\
+ \beta\int_0^1 \psi_t \int_0^t g(t-s)\psi_{xx}(s)dsdx = -\delta\int_0^1 u_{xt}^2 dx.
\end{aligned}
\tag{6.7}
$$

The last term in the left-hand side of (6.7) gives

$$
\begin{aligned}
&\int_0^1 \psi_t \int_0^t g(t-s)\psi_{xx}(s)dsdx \\
&\quad = \int_0^1 \psi_{xt} \int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)dsdx - \left(\int_0^t g(s)ds\right)\int_0^1 \psi_{xt}\psi_x dx \quad (6.8)\\
&\quad = \frac{1}{2}\frac{d}{dt}\left[g\circ\psi_x - \left(\int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx\right] + \frac{1}{2}g(t)\int_0^1 \psi_x^2 dx - \frac{1}{2}g'\circ\psi_x.
\end{aligned}
$$

Lemma 6.1 follows by combining (6.5), (6.7) and (6.8). ∎

Lemma 6.2 *Let (φ, ψ, u) be the solution of (6.4). Then the functional*

$$
F_1(t) := -\rho_1 \int_0^1 \varphi_t \varphi dx - \rho_2 \int_0^1 \psi_t \psi dx
$$

satisfies, for any positive constant ε_1, the estimate

$$
\begin{aligned}
F_1'(t) \le -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + \frac{1}{\varepsilon_1}\int_0^1 u_t^2 dx + c\left(1+\varepsilon_1\right)\int_0^1 \psi_x^2 dx \\
+ \left(K+\varepsilon_1\right)\int_0^1 \left(\varphi_x+\psi\right)^2 dx + cg\circ\psi_x.
\end{aligned} \tag{6.9}
$$

Proof. Direct computations, using the first and the second equations in (6.4), yields

$$
\begin{aligned}
F_1'(t) = -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + K\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \alpha\int_0^1 \psi_x^2 dx \\
- \int_0^1 u_t \varphi_x dx - \int_0^1 u_t \psi dx - \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s)dsdx.
\end{aligned}
$$

By using Young's and Poincaré's inequalities, we get, for $\varepsilon_1 > 0$,

$$F_1'(t) \leq -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + K \int_0^1 (\varphi_x + \psi)^2 \, dx + \frac{\varepsilon_1}{2} \int_0^1 \varphi_x^2 dx$$
$$+ \left(2\alpha + \frac{\varepsilon_1}{2}\right) \int_0^1 \psi_x^2 dx + \frac{1}{\varepsilon_1} \int_0^1 u_t^2 dx + \frac{1}{4\alpha} \int_0^1 \left(\int_0^t g(t-s)\psi_x(s) ds \right)^2 dx. \tag{6.10}$$

The fourth term in the right-hand side of (6.10) gives

$$\int_0^1 \varphi_x^2 dx = \int_0^1 [(\varphi_x + \psi) - \psi]^2 \, dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi^2 dx,$$

and by Poincaré's inequality, we obtain

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx. \tag{6.11}$$

By using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ and the Cauchy-Schwarz inequality, we estimate the last term in the right-hand side of (6.10) as follows:

$$\int_0^1 \left(\int_0^t g(t-s)\psi_x(s) ds \right)^2 dx$$
$$\leq 2 \int_0^1 \left(\int_0^t g(t-s) \left(\psi_x(s) - \psi_x(t)\right) ds \right)^2 dx + 2 \left(\int_0^t g(s) ds \right)^2 \int_0^1 \psi_x^2 dx$$
$$\leq 2 \int_0^t g(s) ds \int_0^1 \int_0^t g(t-s) \left(\psi_x(s) - \psi_x(t)\right)^2 ds dx + 2 \left(\int_0^t g(s) ds \right)^2 \int_0^1 \psi_x^2 dx.$$

By using (A1), we obtain

$$\int_0^1 \left(\int_0^t g(t-s)\psi_x(s) ds \right)^2 dx \leq c \left(g \circ \psi_x + \int_0^1 \psi_x^2 dx \right). \tag{6.12}$$

By substituting (6.11) and (6.12) into (6.10), we get (6.9). ∎

Lemma 6.3 *Let* (φ, ψ, u) *be the solution of* (6.4). *Then the functional*

$$F_2(t) := -\rho_2 \int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx$$

satisfies, for any positive constant ε_2, *the estimate*

$$\begin{aligned} F_2'(t) \leq &- \left(\rho_2 \int_0^t g(s)ds - c\varepsilon_2\right) \int_0^1 \psi_t^2 dx + c\varepsilon_2 \int_0^1 \psi_x^2 dx - \frac{c}{\varepsilon_2} g' \circ \psi_x \\ &+ \varepsilon_2 \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx + c\left(\varepsilon_2 + \frac{1}{\varepsilon_2}\right) g \circ \psi_x. \end{aligned} \tag{6.13}$$

Proof. Taking the derivative of F_2 and using the second equation in (6.4), it easily follows that

$$\begin{aligned} F_2'(t) = \ &\alpha \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx - \rho_2 \left(\int_0^t g(t-s)ds\right) \int_0^1 \psi_t^2 dx \\ &+ K \int_0^1 (\varphi_x + \psi) \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx \\ &- \int_0^1 u_t \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx - \rho_2 \int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))dsdx \\ &- \int_0^1 \left(\int_0^t g(t-s)(\psi_x(s)ds\right) \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx, \end{aligned}$$

where we have used integration by parts and the boundary conditions in (6.4). By using Young's inequality, we obtain, for any $\varepsilon_2 > 0$,

$$
\begin{aligned}
F_2'(t) \leq &-\left(\rho_2 \int_0^t g(s)ds - c\varepsilon_2\right)\int_0^1 \psi_t^2 dx + \varepsilon_2 \int_0^1 (\varphi_x + \psi)\, dx \\
&+ \frac{c}{\varepsilon_2}\int_0^1 \left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\
&+ c\varepsilon_2 \int_0^1 \psi_x^2 dx + \frac{\varepsilon_2}{2}\int_0^1 \left(\int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&+ \frac{c}{\varepsilon_2}\int_0^1 \left(\int_0^t g'(t-s)(\psi(t)-\psi(s))ds\right)^2 dx + \varepsilon_2 \int_0^1 u_t^2 dx \\
&+ \frac{c}{\varepsilon_2}\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx.
\end{aligned} \tag{6.14}
$$

By exploiting the properties of g, Cauchy-Schwarz and Poincaré's inequalities, we get

$$
\begin{aligned}
&\int_0^1 \left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\
&\qquad \leq \int_0^t g(s)ds \int_0^1 \int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)^2 dsdx \\
&\qquad \leq (\alpha - l) g \circ \psi \leq c g \circ \psi_x,
\end{aligned} \tag{6.15}
$$

$$
\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \leq c g \circ \psi_x \tag{6.16}
$$

and

$$
\begin{aligned}
&\int_0^1 \left(\int_0^t g'(t-s)(\psi(t)-\psi(s))ds\right)^2 dx \\
&\qquad \leq \left(\int_0^t -g'(s)ds\right)\int_0^1 \int_0^t -g'(t-s)\left(\psi(t)-\psi(s)\right)^2 dsdx \\
&\qquad \leq -g(0) g' \circ \psi \leq -c g' \circ \psi_x.
\end{aligned} \tag{6.17}
$$

The substitution of (6.12), (6.15)–(6.17) into (6.14), gives (6.13). ▮

Lemma 6.4 *Let* (φ, ψ, u) *be the solution of* (6.4)*. Then the functional*

$$F_3(t) := \rho_3 \int_0^1 u_t u dx + \beta \int_0^1 \varphi_x u dx + \frac{\delta}{2} \int_0^1 u_x^2 dx,$$

satisfies, for any positive constant ε_1*, the estimate*

$$\begin{aligned} F_3' \leq -\frac{\kappa}{2} \int_0^1 u_x^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 u_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx \\ + c \int_0^1 \psi_t^2 dx + \varepsilon_1 \int_0^1 (\varphi_x + \psi)^2 dx. \end{aligned} \tag{6.18}$$

Proof. By differentiating F_3 and using the third equation in (6.4), we obtain

$$F_3'(t) \;=\; -\kappa \int_0^1 u_x^2 dx + \rho_3 \int_0^1 u_t^2 dx + \beta \int_0^1 \varphi_x u_t dx - \beta \int_0^1 \psi_t u dx.$$

By using Young's and Poincaré's inequalities, we obtain, for any $\varepsilon_1 > 0$,

$$F_3'(t) \;\leq\; -\frac{\kappa}{2} \int_0^1 u_x^2 dx + \left(\rho_3 + \frac{\beta^2}{2\varepsilon_1}\right) \int_0^1 u_t^2 dx + \frac{\varepsilon_1}{2} \int_0^1 \varphi_x^2 dx + \frac{\beta^2}{2\kappa} \int_0^1 \psi_t^2 dx.$$

The conclusion of Lemma 6.4 follows courtesy (6.11). ▮

As in [76], we introduce the multiplier w, which is the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0. \tag{6.19}$$

Lemma 6.5 *The solution of* (6.19) *satisfies*

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx \le \int_0^1 \psi_x^2 dx \tag{6.20}$$

and

$$\int_0^1 w_t^2 dx \le \int_0^1 w_{tx}^2 dx \le \int_0^1 \psi_t^2 dx. \tag{6.21}$$

Proof. We multiply (6.19) by w and then integrate over $(0, 1)$ to obtain

$$\int_0^1 w_x^2 dx = \int_0^1 \psi w_x dx.$$

By applying Young's inequality, we get

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx.$$

The last part of (6.20) follows thanks to Poincaré's inequality.

For (6.21), we differentiate (6.19) with respect to t, multiply the resulting equation by w_t and then integrating over $(0, 1)$ to obtain

$$\int_0^1 w_{tx}^2 dx = \int_0^1 \psi_t w_{tx} dx.$$

It easily follows by Young's inequality that

$$\int_0^1 w_{tx}^2 dx \le \int_0^1 \psi_t^2 dx,$$

then by using Poincaré's inequality, we obtain

$$\int_0^1 w_t^2 dx \le \int_0^1 w_{tx}^2 dx \le \int_0^1 \psi_t^2 dx,$$

which complete the proof of Lemma 6.5. ∎

Remark 6.2 It can easily be seen that the solution of (6.19) is explicitly given as

$$w(x,t) = -\int_0^x \psi(s,t)ds + x\int_0^1 \psi(s,t)ds.$$

Lemma 6.6 *Let* (φ, ψ, u) *be the solution of* (6.4). *Then the functional*

$$F_4(t) := \rho_1 \int_0^1 \varphi_t w dx + \rho_2 \int_0^1 \psi_t \psi dx$$

satisfies, for any positive constant ε_3, *the estimate*

$$F_4'(t) \le -\frac{l}{2}\int_0^1 \psi_x^2 dx + c\int_0^1 u_t^2 dx + c\left(1+\frac{1}{\varepsilon_3}\right)\int_0^1 \psi_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + cg\circ\psi_x. \tag{6.22}$$

Proof. A simple differentiation of F_4, then using the first and second equations in (6.4) lead to

$$\begin{aligned} F_4'(t) =& K\int_0^1 w_x^2 dx + \int_0^1 u_t w_x dx + \rho_1 \int_0^1 \varphi_t w_t dx + \int_0^1 u_t \psi dx - \alpha \int_0^1 \psi_x^2 dx \\ & - K\int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s) ds dx, \end{aligned}$$

where we have used integration by parts, (6.19) and the boundary conditions in (6.4). By Young's and Poincaré's inequalities, we get, for any $\varepsilon_3, \varepsilon_4 > 0$,

$$
\begin{aligned}
F_4'(t) \leq & (K+\varepsilon_4)\int_0^1 w_x^2 dx + \frac{1}{2\varepsilon_4}\int_0^1 u_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_3}\int_0^1 w_t^2 dx - K\int_0^1 \psi^2 dx \\
& + \left(2\varepsilon_4 + \int_0^t g(s)ds - \alpha\right)\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\
& + \int_0^1 \left(\int_0^t g(t-s)(\psi_x(s) - \psi_x(t))ds\right)^2 dx.
\end{aligned}
$$

By exploiting (A1), (6.16) and Lemma 6.5 we get

$$
\begin{aligned}
F_4'(t) \leq & -(l-3\varepsilon_4)\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_4}\int_0^1 u_t^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + \left(\rho_2 + \frac{\rho_1^2}{4\varepsilon_3}\right)\int_0^1 \psi_t^2 dx \\
& + \frac{\alpha - l}{4\varepsilon_4} g \circ \psi_x.
\end{aligned}
\tag{6.23}
$$

By setting $\varepsilon_4 = \frac{l}{6}$, the conclusion of our proof follows. ∎

Lemma 6.7 *Let (φ, ψ, u) be the solution of (6.4). Then, the functional*

$$
F_5(t) := \rho_2 \int_0^1 \psi_t(\varphi_x + \psi)dx + \frac{\alpha\rho_1}{K}\int_0^1 \varphi_t \psi_x dx - \frac{\rho_1}{K}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)dsdx
$$

satisfies, for any positive constant ε_1, the estimate

$$
\begin{aligned}
F_5'(t) \leq & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2 \int_0^1 \psi_t^2 dx + c\varepsilon_1 g \circ \psi_x \\
& - \frac{K}{2}\int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_1 \int_0^1 \varphi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 u_{xt}^2 dx \\
& - \frac{c}{\varepsilon_1} g' \circ \psi_x + \left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx + c\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \psi_x^2 dx.
\end{aligned}
\tag{6.24}
$$

Proof. By using equations (6.4) and integrating by parts, we get

$$
\begin{aligned}
F_5'(t) = & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx \\
& - K\int_0^1 (\varphi_x+\psi)^2 dx + \frac{1}{K}\int_0^1 u_{xt}\int_0^t g(t-s)\psi_x(s)dsdx \\
& - \frac{\rho_1 g(0)}{K}\int_0^1 \varphi_t\psi_x dx - \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^t g'(t-s)\psi_x(s)dsdx \\
& + \left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx - \frac{\alpha}{K}\int_0^1 u_{xt}\psi_x dx + \int_0^1 u_t(\varphi_x+\psi)dx \\
= & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx + \int_0^1 u_t(\varphi_x+\psi)dx \\
& - K\int_0^1 (\varphi_x+\psi)^2 dx + \frac{1}{K}\int_0^1 u_{xt}\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))dsdx \\
& - \frac{\rho_1 g(t)}{K}\int_0^1 \varphi_t\psi_x dx + \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))dsdx \\
& + \left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx - \frac{\alpha}{K}\int_0^1 u_{xt}\psi_x dx + \frac{1}{K}\int_0^t g(s)ds\int_0^1 u_{xt}\psi_x dx.
\end{aligned}
$$

By using Young's and Poincaré's inequalities, we get

$$
\begin{aligned}
F_5'(t) \leq & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx \\
& - \frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx + \left(\frac{K}{2} + \frac{\alpha}{2K} + \frac{1}{4\varepsilon_1 K^2} + \frac{1}{2K}\int_0^t g(s)ds\right)\int_0^1 u_{xt}^2 dx \\
& + \varepsilon_1\int_0^1\left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))ds\right)^2 dx + \varepsilon_1\int_0^1 \varphi_t^2 dx \\
& + \left(\frac{(\rho_1 g(t))^2}{2\varepsilon_1 K^2} + \frac{\alpha}{2K} + \frac{1}{2K}\int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx + \left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx \\
& + \frac{\rho_1^2}{2\varepsilon_1 K^2}\int_0^1\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx.
\end{aligned}
$$

Estimate (6.24) is established thanks to (6.16), (6.17) and the properties of g. ∎

In consideration of the boundary terms that appear in (6.24), we define, as in [75], the function

$$m(x) = 2 - 4x, \qquad x \in [0,1].$$

Consequently, we have the following result:

Lemma 6.8 *Let (φ, ψ, u) be the solution of* (6.4)*. Then, for any positive constant ε_1, the functional*

$$F_6(t) := \frac{\varepsilon_1 \rho_1}{K} \int_0^1 m(x)\varphi_t \varphi_x dx + \frac{\rho_2}{4\varepsilon_1} \int_0^1 m(x)\psi_t \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) dx$$

satisfies the estimate

$$\begin{aligned} F_6'(t) \leq &- \left[\varphi_x \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds \right) \right]_{x=0}^{x=1} + c\varepsilon_1 \int_0^1 u_{xt}^2 dx \\ &+ c\varepsilon_1 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{c}{\varepsilon_1} \int_0^1 \psi_t^2 dx + c\varepsilon_1 \int_0^1 \varphi_t^2 dx - \frac{c}{\varepsilon_1} g' \circ \psi_x \\ &+ c \left(\varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} \right) \int_0^1 \psi_x^2 dx + c \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} \right) g \circ \psi_x. \end{aligned} \tag{6.25}$$

Proof. By using equations (6.4) and integrating by parts, we get

$$
\begin{aligned}
F_6'(t) = &-\left[\varepsilon_1\varphi_x^2(1) + \frac{1}{4\varepsilon_1}\left(\alpha\psi_x(1) - \int_0^t g(t-s)\psi_x(1,s)ds\right)^2\right] \\
&- \left[\varepsilon_1\varphi_x^2(0) + \frac{1}{4\varepsilon_1}\left(\alpha\psi_x(0) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2\right] \\
&+ \frac{\alpha\rho_2}{2\varepsilon_1}\int_0^1 \psi_t^2 dx + \frac{2\varepsilon_1\rho_1}{K}\int_0^1 \varphi_t^2 dx - \frac{\rho_2 g(t)}{4\varepsilon_1}\int_0^1 m\psi_t\psi_x dx \\
&+ \frac{1}{2\varepsilon_1}\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx + \varepsilon_1\int_0^1 m\psi_x\varphi_x dx \\
&+ \frac{1}{4\varepsilon_1}\int_0^1 mu_t\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx - \frac{\varepsilon_1}{K}\int_0^1 mu_{xt}\varphi_x dx \\
&- \frac{K}{4\varepsilon_1}\int_0^1 m(\varphi_x+\psi)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx \\
&+ \frac{\rho_2}{4\varepsilon_1}\int_0^1 m\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx + 2\varepsilon_1\int_0^1 \varphi_x^2 dx.
\end{aligned} \tag{6.26}
$$

In what follows, we use Young's and Poincaré's inequalities, (6.12), (6.17), properties of g, the fact that $0 \leq m^2(x) \leq 4, \forall x \in [0,1]$ and $(a+b)^2 \leq 2a^2 + 2b^2$.

$$
\begin{aligned}
&\varphi_x(1)\left(\alpha\psi_x(1) - \int_0^t g(t-s)\psi_x(1,s)ds\right) \\
&\qquad \leq \varepsilon_1\varphi_x^2(1) + \frac{1}{4\varepsilon_1}\left(\alpha\psi_x(1) - \int_0^t g(t-s)\psi_x(1,s)ds\right)^2,
\end{aligned} \tag{6.27}
$$

$$
\begin{aligned}
&-\varphi_x(0)\left(\alpha\psi_x(0) - \int_0^t g(t-s)\psi_x(0,s)ds\right) \\
&\qquad \leq \varepsilon_1\varphi_x^2(0) + \frac{1}{4\varepsilon_1}\left(\alpha\psi_x(1) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2,
\end{aligned} \tag{6.28}
$$

$$
-\frac{1}{K}\int_0^1 mu_{xt}\varphi_x dx \leq \frac{1}{2K}\int_0^1 m^2\varphi_x^2 dx + \frac{1}{2K}\int_0^1 u_{xt}^2 dx \leq c\int_0^1 \varphi_x^2 dx + c\int_0^1 u_{xt}^2 dx, \tag{6.29}
$$

$$
\begin{aligned}
-\frac{\rho_2 g(t)}{4}\int_0^1 m\psi_t\psi_x dx &\le \frac{1}{2}\int_0^1 \psi_x^2 dx + \frac{\rho_2^2 g^2(t)}{32}\int_0^1 m^2\psi_t^2 dx \\
&\le c\int_0^1 \psi_x^2 dx + c\int_0^1 \psi_t^2 dx,
\end{aligned} \tag{6.30}
$$

$$
\int_0^1 m\psi_x\varphi_x dx \le c\int_0^1 \varphi_x^2 dx + c\int_0^1 \psi_x^2 dx, \tag{6.31}
$$

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&\quad\le \alpha^2\int_0^1 \psi_x^2 dx + \int_0^1 \left(\int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&\quad\le c\int_0^1 \psi_x^2 dx + cg\circ\psi_x,
\end{aligned} \tag{6.32}
$$

$$
\begin{aligned}
&\frac{1}{4}\int_0^1 mu_t\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right) dx \\
&\quad\le \frac{\varepsilon_5}{8}\int_0^1 m^2 u_t^2 dx + \frac{1}{8\varepsilon_5}\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&\quad\le c\varepsilon_5\int_0^1 u_{xt}^2 dx + \frac{c}{\varepsilon_5}\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_5} g\circ\psi_x,
\end{aligned} \tag{6.33}
$$

$$
\begin{aligned}
&-\frac{K}{4}\int_0^1 m(\varphi_x+\psi)\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right) dx \\
&\quad\le \frac{\varepsilon_5}{8}\int_0^1 m^2(\varphi_x+\psi)^2 dx + \frac{K^2}{8\varepsilon_5}\int_0^1 \left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx \\
&\quad\le c\varepsilon_5\int_0^1 (\varphi_x+\psi)^2 dx + \frac{c}{\varepsilon_5}\int_0^1 \psi_x^2 dx + \frac{c}{\varepsilon_5} g\circ\psi_x
\end{aligned} \tag{6.34}
$$

and

$$
\begin{aligned}
&\frac{\rho_2}{4}\int_0^1 m\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right) dx \\
&\quad\le \frac{\rho_2^2}{32}\int_0^1 m^2\psi_t^2 dx + \frac{1}{2}\int_0^1 \left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\
&\quad\le c\int_0^1 \psi_t^2 dx - cg'\circ\psi_x.
\end{aligned} \tag{6.35}
$$

The subsitution of (6.27)–(6.35) into (6.26) yields

$$F_6'(t) \leq -\left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + c\left(\varepsilon_1 + \frac{\varepsilon_5}{\varepsilon_1}\right)\int_0^1 u_{xt}^2 dx$$
$$+ c\frac{\varepsilon_5}{\varepsilon_1}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{c}{\varepsilon_1}\int_0^1 \psi_t^2 dx + c\varepsilon_1\int_0^1 \varphi_t^2 dx - \frac{c}{\varepsilon_1} g' \circ \psi_x$$
$$+ c\left(\varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1\varepsilon_5}\right)\int_0^1 \psi_x^2 dx + c\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1\varepsilon_5}\right) g\circ\psi_x + c\varepsilon_1\int_0^1 \varphi_x^2 dx.$$

By setting $\varepsilon_5 = \varepsilon_1^2$ and then using (6.11), we obtain (6.25). ∎

6.3 General Decay Result

In this section, which is subdivided into two parts, we state and prove our main results.

6.3.1 Equal Speed of Propagation $\frac{K}{\rho_1} = \frac{\alpha}{\rho_2}$

In this subsection, we state and prove a general decay result in the case of equal wave-speed propagation.

Remark 6.3 For $\frac{K}{\rho_1} = \frac{\alpha}{\rho_2}$, equation (6.24) takes the form

$$\begin{aligned} F_5'(t) \leq & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx \\ & - \frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx + \varepsilon_1\int_0^1 \varphi_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 u_{xt}^2 dx \\ & - \frac{c}{\varepsilon_1} g'\circ\psi_x + c\varepsilon_1 g\circ\psi_x + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \psi_x^2 dx. \end{aligned} \quad (6.36)$$

Next, we define a Lyapunov functional $\mathcal{L}$ equivalent to the first-order energy functional E. For positive constants N, N_1 and N_2 to be chosen appropriately later, we let

$$\mathcal{L}(t) := NE(t) + \frac{1}{8}F_1(t) + N_1 F_2(t) + F_3(t) + N_2 F_4(t) + F_5(t) + F_6(t). \qquad (6.37)$$

Lemma 6.9 *For N large enough, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t), \qquad \forall t \ge 0. \qquad (6.38)$$

Proof. See the proof of Lemma 3.6 (page 51). ∎

Theorem 6.1 *Let (φ, ψ, u) be the solution of* (6.4). *Assume that g satisfies (A1) and (A2) and that*

$$\frac{K}{\rho_1} = \frac{\alpha}{\rho_2}.$$

Then, for any $t_0 > 0$, there exist two positive constants c_0 and c_1 such that

$$E(t) \le c_0 e^{-c_1 \int_{t_0}^{t} \xi(s)ds}, \qquad \forall t \ge t_0. \qquad (6.39)$$

Proof. By differentiating (6.37) and using (6.6), (6.9), (6.13), (6.18), (6.22), (6.25) and (6.36), using Poincaré's inequality and letting $\varepsilon_2 = \frac{\kappa}{4N_1}$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le \; & - \left[\frac{\rho_1}{8} - c\varepsilon_1 - \varepsilon_3 N_2\right] \int_0^1 \varphi_t^2 dx + \left[\frac{\beta N}{2} - c\left(\frac{1}{\varepsilon_1} + N_1^2\right)\right] g' \circ \psi_x \\
& - \left[N\delta - c\left(1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + N_2\right)\right] \int_0^1 u_{xt}^2 dx - \frac{\kappa}{2} \int_0^1 u_x^2 dx \\
& - \left[\frac{lN_2}{2} - c\left(1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3}\right)\right] \int_0^1 \psi_x^2 dx
\end{aligned}$$

$$
\begin{aligned}
- & \left[N_1\rho_2 \int_0^t g(s)ds - \frac{7\rho_2}{8} - c\left(1 + \frac{1}{\varepsilon_1} + \left(1 + \frac{1}{\varepsilon_3}\right) N_2\right)\right] \int_0^1 \psi_t^2 dx \\
+ & \; c\left[1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3} + N_1^2 + N_2\right] g \circ \psi_x - \left[\frac{K}{8} - c\varepsilon_1\right] \int_0^1 (\varphi_x + \psi)^2 dx.
\end{aligned}
$$

Next, we choose ε_1 small enough such that

$$
\mu_1 = \frac{\rho_1}{8} - c\varepsilon_1 > 0 \quad \text{and} \quad \frac{K}{8} - c\varepsilon_1 > 0,
$$

and then N_2 large enough that

$$
\frac{lN_2}{2} - c\left(1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1^3}\right) > 0.
$$

We then select ε_3 so small that

$$
\mu_1 - \varepsilon_3 N_2 > 0.
$$

Meanwhile, using the fact that the function g is positive, continuous and $g(0) > 0$, we have, for any $t \geq t_0 > 0$,

$$
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0.
$$

Now, we choose N_1 large enough such that

$$
N_1\rho_2 g_0 - \frac{\rho_2}{2} - c\left(1 + \frac{1}{\varepsilon_1} + \left(1 + \frac{1}{\varepsilon_3}\right) N_2\right) > 0.
$$

Finally, we choose N large enough such that (6.38) remains valid,

$$N\delta - c\left(1 + \varepsilon_1 + \frac{1}{\varepsilon_1} + N_2\right) > 0 \quad \text{and} \quad \frac{\beta N}{2} - c\left(\frac{1}{\varepsilon_1} + N_1^2\right) > 0.$$

Consequently, by using Poincaré's inequality and (6.5), we obtain

$$\mathcal{L}'(t) \leq -k_0 E(t) + cg \circ \psi_x, \qquad \forall t \geq t_0, \tag{6.40}$$

where k_0 is a positive constant.

By multiplying (6.40) by $\eta(t)$ and using (A2) and (6.6), we arrive at

$$\eta(t)\mathcal{L}'(t) \leq -k_0\eta(t)E(t) - cE'(t), \qquad \forall t \geq t_0,$$

which can be rewritten as

$$[\eta(t)\mathcal{L}(t) + cE(t)]' - \eta'(t)\mathcal{L}(t) \leq -k_0\eta(t)E(t), \ \forall t \geq t_0.$$

Using the fact that $\eta'(t) \leq 0$, we have

$$(\eta(t)\mathcal{L}(t) + cE(t))' \leq -k_0\eta(t)E(t), \ \forall t \geq t_0.$$

By exploiting (6.38), it can easily be shown that

$$\mathcal{R}(t) = \eta(t)\mathcal{L}(t) + cE(t) \sim E(t). \tag{6.41}$$

Consequently, for some positive constant c_1, we obtain

$$\mathcal{R}'(t) \leq -c_1\eta(t)\mathcal{R}(t), \ \forall t \geq t_0. \tag{6.42}$$

A simple integration of (6.42) over (t_0, t) leads to

$$\mathcal{R}(t) \leq \mathcal{R}(0)e^{-c_1\int_{t_0}^{t}\eta(s)ds}, \ \forall t \geq t_0. \tag{6.43}$$

Finally, (6.39) is established by combining (6.41) and (6.43). ▮

Remark 6.4 Estimate (6.39) also holds for $t \in [0, t_0]$ by virtue of continuity and boundedness of E and η. See Remark 3.2 on page 57 for similar proof.

6.3.2 Nonequal Speed of Propagation $\frac{K}{\rho_1} \neq \frac{\alpha}{\rho_2}$

In this subsection, we treat the case of different wave-speed propagation. In this regard, we establish a general decay result, which depends on the asymptotic behaviour of g and the regularity of the initial data. The main theorem in this subsection is:

Theorem 6.2 *Let (φ, ψ, u) be the strong solution of* (6.4)*. Assume that g satisfies (A1) and (A2) and*

$$\frac{K}{\rho_1} \neq \frac{\alpha}{\rho_2}.$$

Then, for any $t_0 > 0$, there exists a positive constant c_2 such that

$$E(t) \leq \frac{c_2}{\int_0^t \eta(s)ds}, \qquad \forall t \geq t_0. \tag{6.44}$$

In order to establish this result, we need the second-order energy associated with problem (6.4). To this end, we differentiate (6.4) with respect to t and use the fact that

$$\begin{aligned}
\frac{d}{dt}\int_0^t g(t-s)\psi_{xx}(x,s)ds &= \frac{d}{dt}\int_0^t g(s)\psi_{xx}(x,t-s)ds \\
&= g(t)\psi_{xx}(x,0) + \int_0^t g(s)\psi_{xxt}(x,t-s)ds \\
&= g(t)\psi_{0xx}(x) + \int_0^t g(t-s)\psi_{xxt}(x,s)ds
\end{aligned}$$

to get the system

$$\begin{cases}
\rho_1\varphi_{ttt} - K(\varphi_{xt} + \psi_t)_x + u_{xtt} = 0, & x \in (0,1),\ t > 0, \\
\rho_2\psi_{ttt} - \alpha\psi_{xxt} + K(\varphi_{xt} + \psi_t) - u_{tt} + g(t)\psi_{0xx}(x) & \\
\qquad + \int_0^t g(t-s)\psi_{xxt}(x,s)ds = 0, & x \in (0,1),\ t > 0, \\
\rho_3 u_{ttt} - \kappa u_{xxt} - \delta u_{xxtt} + \beta\varphi_{xtt} + \beta\psi_{tt} = 0, & x \in (0,1),\ t > 0, \\
\varphi_t(0,t) = \varphi_t(1,t) = \psi_t(0,t) = \psi_t(1,t) = u_t(0,t) = u_t(1,t) = 0, & t \geq 0.
\end{cases} \tag{6.45}$$

The second-order energy is defined by

$$\mathcal{E}(t) = E_1(\varphi_t, u_t, \psi_t), \tag{6.46}$$

where E_1 is given in (6.5).

Now, as in [34], we prove the following lemma:

Lemma 6.10 *Let* (φ, ψ, u) *be the strong solution of* (6.4). *Then there exits* $c > 0$ *such that the energy functional* $\mathcal{E}(t)$, *defined by* (6.46), *satisfies*

$$\mathcal{E}'(t) = -\delta \int_0^1 u_{xtt}^2 dx + \frac{\beta}{2} g' \circ \psi_{xt} - \frac{\beta}{2} g(t) \int_0^1 \psi_{xt}^2 dx - \beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \quad (6.47)$$

and

$$\mathcal{E}(t) \leq c, \qquad \forall t \geq 0. \tag{6.48}$$

Proof. Multiplying the first equation of (6.45) by $\beta\varphi_{tt}$, the second by $\beta\psi_{tt}$, and the third by u_{tt}, integrating over $(0,1)$, and summing up, as in Lemma 6.1, we obtain (6.47).

To prove (6.48), following the idea of [34], we observe that

$$\mathcal{E}'(t) \leq -\beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx, \qquad \forall t \geq 0. \tag{6.49}$$

By using the fact that

$$\frac{\beta}{2} g(t) \int_0^1 \left(\sqrt{\rho_2} \psi_{tt} + \frac{1}{\sqrt{\rho_2}} \psi_{0xx} \right)^2 dx \geq 0, \qquad \forall t \geq 0,$$

we get

$$-\beta g(t) \int_0^1 \psi_{tt} \psi_{0xx}(x) dx \leq \frac{\beta \rho_2}{2} g(t) \int_0^1 \psi_{tt}^2 dx + \frac{\beta}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx. \tag{6.50}$$

The combination of (6.46), (6.49) and (6.50) gives

$$\mathcal{E}'(t) \le g(t)\mathcal{E}(t) + \frac{\beta}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx. \tag{6.51}$$

It is clear from (6.51) that

$$\frac{d}{dt}\left(\mathcal{E}(t)e^{-\int_0^t g(s)ds}\right) \le \frac{\beta}{2\rho_2} g(t) e^{-\int_0^t g(s)ds} \int_0^1 \psi_{0xx}^2 dx \le \frac{\beta}{2\rho_2} g(t) \int_0^1 \psi_{0xx}^2 dx.$$

A simple integration yields

$$\mathcal{E}(t)e^{-\int_0^t g(s)ds} \le \mathcal{E}(0) + \frac{\beta}{2\rho_2}\left(\int_0^t g(s)ds\right)\int_0^1 \psi_{0xx}^2 dx.$$

Using (A1), it follows that

$$\mathcal{E}(t)e^{-\int_0^\infty g(s)ds} \le \mathcal{E}(0) + \frac{\beta(\alpha - l)}{2\rho_2}\int_0^1 \psi_{0xx}^2 dx.$$

Consequently. we obtain

$$\mathcal{E}(t) \le c, \qquad \forall t \ge 0.$$

∎

Lemma 6.11 *Let (φ, ψ, u) be the strong solution of (6.4), then for all $t \ge t_0$, we have*

$$\left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx \le \varepsilon_1 \int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_1}\Big(g(t) - g' \circ \psi_x + g \circ \psi_{xt}\Big). \tag{6.52}$$

Proof. Similar to [72], we consider

$$\left(\rho_2 - \frac{\alpha\rho_1}{K}\right)\int_0^1 \varphi_{xt}\psi_t dx = \left(\frac{\alpha\rho_1}{K} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt} dx$$
$$= \frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))dsdx$$
$$+ \frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx. \tag{6.53}$$

We estimate the terms in the right-hand side of (6.53) as follows:

By using Young's inequality, for any $\varepsilon_1 > 0$, the first term gives

$$\frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))dsdx$$
$$\leq \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx + \frac{1}{2\varepsilon_1}\left(\frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\right)^2 \int_0^1 \left(\int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))ds\right)^2 dx.$$

By exploiting (6.16) for ψ_{xt}, we obtain

$$\frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)(\psi_{xt}(t) - \psi_{xt}(s))dsdx \leq \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_1} g \circ \psi_{xt}. \tag{6.54}$$

By using integration by parts with respect to t, the second term in the right-hand side of (6.53) gives

$$
\begin{aligned}
&\frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \\
&= \frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \left(g(0)\psi_x(t) - g(t)\psi_{0x} + \int_0^t g'(t-s)\psi_x(s)ds\right)dx \\
&= \frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \left(g(t)\psi_x(t) - g(t)\psi_{0x} - \int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)dx.
\end{aligned}
$$

By using Young's inequality, for any $\varepsilon_1 > 0$, we get

$$
\begin{aligned}
&\frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \\
&\qquad \leq \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx + \frac{1}{2\varepsilon_1}\left(\frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\right)^2 \int_0^1 \Big[g(t)\psi_x(t) - g(t)\psi_{0x} \\
&\qquad\qquad - \int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\Big]^2 dx,
\end{aligned}
$$

By using the fact that $(a-b)^2 \leq 2a^2 + 2b^2$, we get

$$
\begin{aligned}
&\frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \leq \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx \\
&\quad + \frac{c}{\varepsilon_1}g^2(t)\int_0^1 \left(\psi_x^2(t)+\psi_{0x}^2\right)dx + \frac{c}{\varepsilon_1}\int_0^1 \left(\int_0^t g'(t-s)\left(\psi_x(t)-\psi_x(s)\right)ds\right)^2 dx.
\end{aligned}
$$

By exploiting (6.17), we obtain

$$
\begin{aligned}
&\frac{\left(\frac{\alpha\rho_1}{K}-\rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \\
&\qquad \leq \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_1}g^2(t)\int_0^1 \left(\psi_x^2(t)+\psi_{0x}^2\right)dx - \frac{c}{\varepsilon_1}g' \circ \psi_x,
\end{aligned}
\tag{6.55}
$$

From (6.5), it is clear that

$$\left(\alpha - \int_0^t g(s)ds\right)\int_0^1 \psi_x^2 dx \le \frac{2}{\beta}E(t),$$

and by using the fact that $\alpha - \int_0^t g(s)ds > 0$, thanks to (A1) and the non-increasingness of E, we get

$$\int_0^1 \psi_x^2 dx \le \frac{2}{\beta\left(\alpha - \int_0^t g(s)ds\right)}E(t) \le \frac{2}{\beta\left(\alpha - \int_0^\infty g(s)ds\right)}E(t) \le cE(0). \quad (6.56)$$

By substituting (6.56) into (6.55) and using the fact that g is bounded, we obtain

$$\frac{\left(\frac{\alpha\rho_1}{K} - \rho_2\right)}{\int_0^t g(s)ds}\int_0^1 \varphi_t \int_0^t g(t-s)\psi_{xt}(s)dsdx \le \frac{\varepsilon_1}{2}\int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_1}g(t) - \frac{c}{\varepsilon_1}g' \circ \psi_x. \quad (6.57)$$

The combination of (6.54) and (6.57) gives (6.52), which is the desired result. ∎

Remark 6.5 For $\frac{K}{\rho_1} \neq \frac{\alpha}{\rho_2}$, taking Lemma 6.11 into account, equation (6.24) takes the form

$$\begin{aligned} F_5'(t) \le & \left[\varphi_x\left(\alpha\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon_1}g(t) \\ & + 2\varepsilon_1\int_0^1 \varphi_t^2 dx - \frac{K}{2}\int_0^1(\varphi_x+\psi)^2dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 u_{xt}^2 dx \\ & + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \psi_x^2 dx - \frac{c}{\varepsilon_1}g' \circ \psi_x + c\varepsilon_1 g \circ \psi_x + \frac{c}{\varepsilon_1}g \circ \psi_{xt}. \end{aligned} \quad (6.58)$$

Proof of Theorem 6.2. To finalize the proof of Theorem 6.2, we use the same Lyapunov functional $\mathcal{L}$ defined in (6.37). That is

$$\mathcal{L}(t) := NE(t) + \frac{1}{8}F_1(t) + N_1F_2(t) + F_3(t) + N_2F_4(t) + F_5(t) + F_6(t),$$

but we use (6.58) instead of (6.24). Following the same steps with the same choice of the constants (up to (6.40)) as in the proof of Theorem 6.1, we obtain, for all $t \geq t_0$,

$$\mathcal{L}'(t) \leq -k_0E(t) + cg \circ \psi_x + cg \circ \psi_{xt} + cg(t). \tag{6.59}$$

By multiplying (6.59) by $\eta(t)$ and using (A2), we obtain

$$\eta(t)\mathcal{L}'(t) \leq -k_0\eta(t)E(t) - cg' \circ \psi_x - cg' \circ \psi_{xt} - cg'(t), \qquad \forall t \geq t_0,$$

that is

$$\eta(t)E(t) \leq -k_2\eta(t)\mathcal{L}'(t) - cg' \circ \psi_x - cg' \circ \psi_{xt} - cg'(t), \qquad \forall t \geq t_0, \tag{6.60}$$

where $k_2 = \frac{1}{k_0}$. Integrating (6.60) over $[t_0, t]$, we get

$$\begin{aligned}\int_{t_0}^{t} \eta(s)E(s)ds \leq k_2 &\left[\eta(t_0)\mathcal{L}(t_0) - \eta(t)\mathcal{L}(t) + \int_{t_0}^{t} \eta'(s)\mathcal{L}(s)ds\right] \\ &- c\int_{t_0}^{t} g' \circ \psi_x(s)ds - c\int_{t_0}^{t} g' \circ \psi_{xt}(s)ds + c, \quad \forall t \geq t_0.\end{aligned} \tag{6.61}$$

We estimate each terms in (6.61) as follows:

Since η is positive and nonincreasing, and $\mathcal{L}$ is positive and equivalent to E (recall that E is positive and nonincreasing) , so

$$\eta(t_0)\mathcal{L}(t_0) \leq cE(0) \leq c, \qquad \forall t \geq t_0, \tag{6.62}$$

$$-\eta(t)\mathcal{L}(t) + \int_{t_0}^{t} \eta'(s)\mathcal{L}(s)ds \leq 0, \qquad \forall t \geq t_0, \tag{6.63}$$

Using (6.6), we get

$$-\int_{t_0}^{t} g' \circ \psi_x(s)ds \leq -\frac{2}{\beta}\int_{t_0}^{t} E'(s)ds = \frac{2}{\beta}\big(E(t_0) - E(t)\big) \leq cE(0) \leq c, \ \forall t \geq t_0. \tag{6.64}$$

Similarly, by using (6.47) and (6.48), we obtain

$$\begin{aligned} -\int_{t_0}^{t} g' \circ \psi_{xt}(s)ds \leq& -\frac{2}{\beta}\int_{t_0}^{t} \mathcal{E}'(s)ds - 2\int_{t_0}^{t} g(s)\int_0^1 \psi_{tt}\psi_{0xx}dxds \\ \leq& c - 2\int_{t_0}^{t} g(s)\int_0^1 \psi_{tt}\psi_{0xx}dxds, \qquad \forall t \geq t_0. \end{aligned} \tag{6.65}$$

By using Young's inequality, the last term in (6.65) gives

$$-\int_0^1 \psi_{tt}\psi_{0xx}(x)dx \leq \frac{1}{2}\int_0^1 \psi_{tt}^2 dx + \frac{1}{2}\int_0^1 \psi_{0xx}^2 dx. \tag{6.66}$$

By exploiting (6.46) and (6.48), it follows that

$$\int_0^1 \psi_{tt}^2 dx \leq \frac{2}{\rho_2 \beta}\mathcal{E}(t) \leq c.$$

Thus, (6.66) yields

$$-\int_0^1 \psi_{tt}\psi_{0xx}(x)dx \leq c. \tag{6.67}$$

By subsituting (6.67) into (6.65), we get

$$-\int_{t_0}^{t} g' \circ \psi_{xt}(s) \leq c + c\int_{t_0}^{t} g(s)ds, \qquad \forall t \geq t_0,$$

and by $(A1)$, we obtain

$$-\int_{t_0}^{t} g' \circ \psi_{xt}(s) \leq c + c(\alpha - l) \leq c, \qquad \forall t \geq t_0. \tag{6.68}$$

Consequently, using (6.62)–(6.64) and (6.68), we get

$$\int_{t_0}^{t} \eta(s)E(s)ds \leq c, \quad \forall t \geq t_0. \tag{6.69}$$

Since E is nonincreasing, then

$$\begin{aligned} E(t)\int_0^t \eta(s)ds \leq \int_0^t \eta(s)E(s)ds = \int_0^{t_0} \eta(s)E(s)ds + \int_{t_0}^{t} \eta(s)E(s)ds \\ \leq t_0\eta(0)E(0) + c \leq c_2 \qquad \forall t \geq t_0. \end{aligned}$$

Therefore,

$$E(t) \leq \frac{c_2}{\int_0^t \eta(s)ds} \qquad \forall t \geq t_0,$$

which is the conclusion of Theorem 6.2.

CHAPTER 7

TIMOSHENKO-THERMOELASTIC SYSTEM WITH SECOND SOUND AND A DELAY TERM

In this chapter, we consider a Timoshenko-Thermoelastic system with second sound and delay. More precisely, we study the following one-dimensional problem

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x+\psi\right)_x + \mu\varphi_t(x,t-\tau_0) = 0, & x\in(0,1),\ t>0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x+\psi\right) + \delta\theta_x = 0, & x\in(0,1),\ t>0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & x\in(0,1),\ t>0, \\ \tau q_t + \beta q + \theta_x = 0, & x\in(0,1),\ t>0, \end{cases} \tag{7.1}$$

together with the following the initial, boundary and history conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x) & x \in (0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ q(x,0) = q_0(x), & x \in (0,1), \\ \varphi_t(x,-t) = f_0(x,t), & x \in (0,1),\ t \in (0,\tau_0), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0,\ t \geq 0, \end{cases} \tag{7.2}$$

where φ is the transverse displacement of the beam, ψ is the rotation angle of the filament, θ is the difference temperature, q is the heat flux, the coefficients $\rho_i,\ \beta, K,\ \delta, b, \tau$ are positive constants, μ is a real number, and $\tau_0 > 0$ represents the time delay. This is a thermoelastic system of Timoshenko type with a delay where the heat flux is given by Cattaneo's law. The system is subjected to a constant internal delay, boundary conditions of Neumann-Dirichlect type, initial conditions $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0$ and history function f_0. We investigate system (7.1)-(7.2) and establish an exponential decay result under a smallness condition on the delay and the stability number introduced first by Santos *et al* in [108]. Furthermore, in the absence of a delay, we prove the polynomial decay result using multiplier method instead of the semigroup method used in [108].

7.1 Introduction

In 1921, Timoshenko [115] developed a simple model describing the transverse vibration of a beam. The model is given by a system of coupled hyperbolic equations of the form

$$
\begin{cases}
\rho u_{tt} = \left(K\left(u_x - \varphi\right)\right)_x, & \text{in } (0,L)\times(0,\infty), \\
I_\rho \varphi_{tt} = \left(EI\varphi_x\right)_x + K\left(u_x - \varphi\right), & \text{in } (0,L)\times(0,\infty),
\end{cases}
\tag{7.3}
$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

The model has been studied by a great number of researchers and various damping mechanisms have been utilized to stabilize the vibrations. The obtained results in those work show that the presence of damping terms in both equations leads to uniform stability (exponential or polynomial) regardless of the values of the constants ρ, I_ρ, E, I and K. This has been demonstrated by Kim and Renardy [43], Messaoudi and Mustafa [61] and others.

In the case of only one damping in the second equation of (7.3), uniform stability is obtained for weak solutions if $\dfrac{K}{\rho} = \dfrac{EI}{I_\rho}$. Whereas, in the opposite case $\left(\dfrac{K}{\rho} \neq \dfrac{EI}{I_\rho}\right)$, a weaker rate of decay is obtained for strong solutions. In this regard, we quote, among others, the work of Soufyane and Wehbe [110], Guesmia and Messaoudi [26, 27], Rivera and Fernández Sare [77], Rivera and Racke [78], Messaoudi and Mustafa [64, 67], Messaoudi and Said-Houari [66].

For stabilization of Timoshenko systems via heat effect, Rivera and Racke [75] considered the following system

$$\begin{cases} \rho_1 u_{tt} - \sigma\left(u_x, \varphi\right)_x = 0, \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k\left(u_x + \varphi\right) + \gamma\theta_x = 0, \\ \rho_3 \theta_t - k\theta_{xx} + \gamma\varphi_{tx} = 0. \end{cases} \tag{7.4}$$

Under appropriate conditions of $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for linearized system in the case of equal wave speeds and non exponential stability for the case of different wave speeds. Various general stability estimates for system (7.4) in the linear case were proved in [30] by adding an infinite memory on the first or second equation. These estimates depend on the regularity of the initial data and the speeds of wave propagation and allow the kernel to have a weak decay at infinity, which can be arbitrarily close to $\frac{1}{t}$.

Concerning second sound, Messaoudi *et al.* [65] studied the following problem

$$\begin{cases} \rho_1 u_{tt} - \sigma\left(u_x, \varphi\right)_x + \mu u_t = 0, \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k\left(u_x + \varphi\right) + \beta\theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta\varphi_{tx} = 0, \\ \tau q_t + q + \kappa\theta_x = 0, \end{cases}$$

where $(x,t) \in (0,L) \times (0,\infty)$, and established several exponential decay results for both linear and nonlinear cases under appropriate conditions on the nonlinear function σ.

In 2009, Fernández Sare and Racke [19] looked into the system

$$\begin{cases} \rho_1 u_{tt} - K\left(u_x + \varphi\right)_x = 0, & \text{in } (0,1)\times(0,\infty) \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + \displaystyle\int_0^\infty g(s)\varphi_{xx}\left(.,t-s\right)ds + K\left(u_x+\varphi\right) + \beta\theta_x = 0, & \text{in } (0,1)\times(0,\infty) \\ \rho_3\theta_t + Kq_x + \beta\varphi_{tx} = 0, & \text{in } (0,1)\times(0,\infty) \\ \tau q_t + q + K\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{7.5}$$

for $g \equiv 0$ and for $g > 0$. They proved in both cases that (7.5) is no longer exponentially stable even if the propagation speeds are equal $\left(\dfrac{K}{\rho_1} = \dfrac{b}{\rho_2}\right)$ and g is of exponential decay. The results of [19] were generalized in [30] to the case where g does not converge exponentially to zero. On the other hand, it was proved in [30] that the uniform stability (exponential, polynomial or others depending on the growth of g at infinity) holds without any restriction on the parameters if the infinite memory is considered in the first equation of (7.5).

Very recently, Santos *et al*. [108] considered (7.5) for $g = 0$ and introduced a new stability number

$$\chi = \left(\tau - \frac{\rho_1}{K\rho_3}\right)\left(\rho_2 - \frac{b\rho_1}{K}\right) - \frac{\tau\rho_1\delta^2}{K\rho_3}$$

and used the semigroup method to obtain exponential decay result for $\chi = 0$, and polynomial decay for $\chi \neq 0$ provided that $\tau - \dfrac{\rho_1}{K\rho_3} > 0$. A similar result was obtained by Said-Houari and Kasimov in [107] by considering the Cauchy problem for the one-dimensional Timoshenko system coupled with heat conduction governed by either the Cattaneo law or the Fourier law. They proved that heat dissipation alone is sufficient to stabilize the system in both cases and then concluded that the Timoshenko-Fourier and the Timoshenko-Cattaneo systems have the same decay rate, which depends on a certain stability number (which is a function of the parameters of the system) as identified previously by Santos *et al.* in [108] for Timoshenko system in a bounded domain.

Introducing a delay term in the internal feedback of Timoshenko system with second sound makes the problem considered in this chapter different from those considered so far in the literature. We refer the reader to [45], [104]–[106] for some related results concerning Timoshenko type system with delay.

The rest of the chapter is organized as follows. The well-posedness of the problem is considered in section 7.2. We use the multiplier method to establish the exponential decay of the energy in section 7.3. In section 7.4, we reproduce the polynomial decay of [108] using the multiplier method instead of the semigroup method, in the case of absence of delay.

7.2 The Well-posedness of the Problem

In this section, we give the existence and uniqueness result for problem (7.1)-(7.2) using the semigroup theory. To this end, we first transform (7.1) into an equivalent problem by introducing, as in all other chapters, a new dependent variable

$$z(x,\rho,t) = \varphi_t(x, t-\rho\tau_0), \qquad x \in (0,1),\ \rho \in (0,1),\ t > 0.$$

A simple differentiation shows that z satisfies

$$\tau_0 z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad x \in (0,1),\ \rho \in (0,1),\ t > 0.$$

Consequently, problem (7.1)-(7.2) is equivalent to the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu z(x,1,t) = 0, & x \in (0,1),\ t > 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & x \in (0,1),\ t > 0, \\ \rho_3 \theta_t + q_x + \delta\psi_{tx} = 0, & x \in (0,1),\ t > 0, \\ \tau q_t + \beta q + \theta_x = 0, & x \in (0,1),\ t > 0, \\ \tau_0 z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in (0,1),\ \rho \in (0,1),\ t > 0, \\ z(x,0,t) = \varphi_t(x,t), & x \in (0,1),\ t > 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \theta(x,0) = \theta_0(x) & x \in (0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ q(x,0) = q_0(x), & x \in (0,1), \\ z(x,\rho,0) = f_0(x,\rho\tau_0), & x \in (0,1),\ \rho \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \forall t \geq 0. \end{cases} \tag{7.6}$$

Thus, we shall consider problem (7.6) instead of (7.1)-(7.2).

From $(7.6)_2, (7.6)_4$ and the boundary conditions, we easily verify that

$$\frac{d^2}{dt^2}\left(\int_0^1 \psi(x,t)\right) + \frac{K}{\rho_2}\int_0^1 \psi(x,t) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\int_0^1 q(x,t)\right) + \frac{\beta}{\tau}\int_0^1 q(x,t) = 0.$$

So, if we set

$$\overline{\psi}(x,t) = \psi(x,t) - \left(\int_0^1 \psi_0(x)dx\right)\cos\sqrt{\frac{K}{\rho_2}}t - \sqrt{\frac{\rho_2}{K}}\left(\int_0^1 \psi_1(x)dx\right)\sin\sqrt{\frac{K}{\rho_2}}t$$

and

$$\overline{q}(x,t) = q(x,t) - \left(\int_0^1 q_0(x)dx\right)\exp\left(-\frac{\beta t}{\tau}\right),$$

then simple substitution shows that $(\varphi, \overline{\psi}, \theta, \overline{q}, z)$ satisfies equations and the boundary condition in (7.6). More importantly

$$\int_0^1 \overline{\psi}(x,t)dx = 0 \quad \text{and} \quad \int_0^1 \overline{q}(x,t)dx = 0, \quad \forall t \geq 0.$$

Hence, the use of Poincaré's inequality for $\overline{\psi}$ is justified. From now on, we work with $\overline{\psi}$ and $\overline{q}$ but write ψ and q for convenience. Introducing the vector function $\Phi = (\varphi, u, \psi, v, \theta, q, z)^T$, where $u = \varphi_t$ and $v = \psi_t$, system (7.6) can be re-written as

$$\begin{cases} \Phi'(t) + (\mathcal{A} + \mathcal{B})\Phi(t) = 0, & t > 0, \\ \Phi(0) = \Phi_0 = (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ \theta_0,\ q_0,\ f_0)^T, \end{cases} \tag{7.7}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -u \\ -\dfrac{K}{\rho_1}\left(\varphi_x+\psi\right)_x + \dfrac{|\mu|}{\rho_1}u + \dfrac{\mu}{\rho_1}z(.,1) \\ -v \\ -\dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}\left(\varphi_x+\psi\right) + \dfrac{\delta}{\rho_2}\theta_x \\ \dfrac{1}{\rho_3}q_x + \dfrac{\delta}{\rho_3}v_x \\ \dfrac{\beta}{\tau}q + \dfrac{1}{\tau}\theta_x \\ \frac{1}{\tau_0}z_\rho \end{pmatrix}$$

and the operator $\mathcal{B} : D(\mathcal{B}) = \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{B}\Phi = \frac{|\mu|}{\rho_1}\begin{pmatrix} 0 \\ -u \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We equally introduce

$$L^2_\star(0,1) = \{w \in L^2(0,1) \ : \int_0^1 w(s)ds = 0\}, \qquad H^1_\star(0,1) = H^1(0,1) \cap L^2_\star(0,1)$$

and

$$H^2_\star(0,1) = \{w \in H^2(0,1) \ : w_x(0) = w_x(1) = 0\}$$

and the Hilbert space

$$\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1) \times L^2(0,1) \times L^2_\star(0,1) \times L^2\big((0,1), L^2(0,1)\big),$$

equipped with the inner product

$$\begin{aligned}(\Phi, \tilde{\Phi})_{\mathcal{H}} =& K\int_0^1 (\varphi_x + \psi)\left(\tilde{\varphi}_x + \tilde{\psi}\right) dx + \rho_1 \int_0^1 u\tilde{u}dx + b\int_0^1 \psi_x \tilde{\psi}_x dx + \rho_2 \int_0^1 v\tilde{v}dx \\ &+ \rho_3 \int_0^1 \theta\tilde{\theta}dx + \tau \int_0^1 q\tilde{q}dx + \tau_0|\mu| \int_0^1 \int_0^1 z\tilde{z}d\rho dx.\end{aligned}$$

The domain of $\mathcal{A}$ is then

$$D(\mathcal{A}) = \left\{ \begin{array}{ccc} \Phi \in \mathcal{H} \mid \varphi \in H^2(0,1) \cap H^1_0(0,1), & \psi \in H^2_\star(0,1) \cap H^1_\star(0,1), & u, \theta \in H^1_0(0,1), \\ v, q \in H^1_\star(0,1), & z_\rho \in L^2\left((0,1), L^2(0,1)\right), & z(x,0) = u(x) \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result:

Theorem 7.1 *Let $\Phi_0 \in \mathcal{H}$, then there exists a unique solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem* (7.7)*. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. We use the semigroup approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator and that $\mathcal{B}$ is a Lipschitz continuous operator. In what follows we prove that $\mathcal{A}$ is monotone. For any $\Phi \in D(\mathcal{A})$, we have

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = |\mu| \int_0^1 u^2 dx + \beta \int_0^1 q^2 dx + \mu \int_0^1 uz(.,1) dx + |\mu| \int_0^1 \int_0^1 zz_\rho d\rho dx. \quad (7.8)$$

By using Young's inequality, the third term in the right-hand side of (7.8) gives

$$-\mu \int_0^1 uz(.,1)dx \leq \frac{|\mu|}{2} \int_0^1 z^2(.,1)dx + \frac{|\mu|}{2} \int_0^1 u^2 dx,$$

which implies that

$$\mu \int_0^1 uz(.,1)dx \geq -\frac{|\mu|}{2} \int_0^1 z^2(.,1)dx - \frac{|\mu|}{2} \int_0^1 u^2 dx.$$

Also, using integration by parts and the fact that $z(x, 0) = u(x)$, the last term in the right-hand side of (7.8) gives

$$\int_0^1 \int_0^1 zz_\rho d\rho dx = \frac{1}{2} \int_0^1 z^2(.,1)dx - \frac{1}{2} \int_0^1 u^2 dx.$$

Consequently, (7.8) yields

$$(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \geq \beta \int_0^1 q^2 dx.$$

Hence $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective. Given $\mathcal{G} = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{H}$, we prove that there exists $\Phi \in D(\mathcal{A})$ satisfying

$$(I+\mathcal{A})\,\Phi=\mathcal{G}. \tag{7.9}$$

That is,

$$\begin{cases} -u+\varphi=g_1, \\ -K\left(\varphi_x+\psi\right)_x+\left(|\mu|+\rho_1\right)u+\mu z(.,1)=\rho_1 g_2, \\ -v+\psi=g_3, \\ -b\psi_{xx}+K\left(\varphi_x+\psi\right)+\delta\theta_x+\rho_2 v=\rho_2 g_4, \\ q_x+\delta v_x+\rho_3\theta=\rho_3 g_5, \\ \left(\beta+\tau\right)q+\theta_x=\tau g_6, \\ z_\rho+\tau_0 z=\tau_0 g_7. \end{cases} \tag{7.10}$$

Suppose $\varphi,\ \psi$ and q are given with the appropriate regularity, then $(7.10)_1, (7.10)_3$ and $(7.10)_6$ yield

$$u=\varphi-g_1\ \in H_0^1(0,1), \tag{7.11}$$

$$v=\psi-g_3\ \in H_\star^1(0,1) \tag{7.12}$$

and

$$\theta_x=\tau g_6-\left(\beta+\tau\right)q\ \in L_\star^2(0,1), \tag{7.13}$$

respectively. From (7.13), we define

$$\theta = \tau \int_0^x g_6 dx - (\beta + \tau) \int_0^x q dx,$$

then

$$\theta(0,t) = \theta(1,t) = 0.$$

The seventh equation in (7.10) together with (7.11) and the fact that $z(x,0) = u(x)$ yield

$$z(x,\rho) = \varphi(x) e^{-\tau_0 \rho} - e^{-\tau_0 \rho} g_1(x) + \tau_0 e^{-\tau_0 \rho} \int_0^\rho e^{\tau_0 s} g_7(x,s) ds. \tag{7.14}$$

By using (7.11) – (7.14), it can easily be shown that φ, ψ and q satisfy

$$\begin{cases} -K \left(\varphi_x + \psi\right)_x + \tilde{\mu} \varphi = h_1 \ \in L^2(0,1), \\ -b \psi_{xx} + K \left(\varphi_x + \psi\right) + \rho_2 \psi - \left(\beta + \tau\right) \delta q = h_2 \ \in L^2_\star(0,1), \\ -q_x + \left(\beta + \tau\right) \rho_3 \displaystyle\int_0^x q(y) dy - \delta \psi_x = h_3 \ \in L^2(0,1), \end{cases} \tag{7.15}$$

where

$$\begin{cases} \tilde{\mu} = \rho_1 + |\mu| + \mu e^{-\tau_0}, \\ h_1 = \tilde{\mu} g_1 + \rho_1 g_2 - \mu \tau_0 e^{-\tau_0} \displaystyle\int_0^1 e^{\tau_0 s} g_7(x,s) ds, \\ h_2 = \rho_2 \left(g_3 + g_4\right) - \tau \delta g_6, \\ h_3 = -\delta g_{3x} - \rho_3 \left(g_5 - \tau \displaystyle\int_0^x g_6(y) dy\right). \end{cases}$$

The variational formulation corresponding to (7.15) takes the form

$$B\big((\varphi,\psi,q),(\varphi_1,\psi_1,q_1)\big) = F\big(\varphi_1,\psi_1,q_1\big), \tag{7.16}$$

where $B : \big[H_0^1(0,1)\times H_\star^1(0,1)\times L_\star^2(0,1)\big]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}
B\big((\varphi,\psi,q),(\varphi_1,\psi_1,q_1)\big) &= K\int_0^1 (\varphi_x+\psi)(\varphi_{1x}+\psi_1)\,dx + (\beta+\tau)\int_0^1 qq_1dx \\
&\quad + b\int_0^1 \psi_x\psi_{1x}dx + \rho_2\int_0^1 \psi\psi_1 dx - \delta(\beta+\tau)\int_0^1 q\psi_1 dx \\
&\quad + \tilde{\mu}\int_0^1 \varphi\varphi_1 dx + \delta(\beta+\tau)\int_0^1 \psi q_1 dx \\
&\quad + \rho_3(\beta+\tau)^2\int_0^1\left(\int_0^x q(y)dy\int_0^x q_1(y)dy\right)dx
\end{aligned}$$

and $F : \big[H_0^1(0,1)\times H_\star^1(0,1)\times L_\star^2(0,1)\big] \longrightarrow \mathbb{R}$ is the linear functional given by

$$F\big(\varphi_1,\psi_1,q_1\big) = \int_0^1 h_1\varphi_1dx + \int_0^1 h_2\psi_1dx + \int_0^1 h_3\int_0^x q_1(y)dydx.$$

Now, for $V = H_0^1(0,1)\times H_\star^1(0,1)\times L_\star^2(0,1)$ equipped with the norm

$$\|(\varphi,\psi,q)\|_V = \|(\varphi_x+\psi)\|_2^2 + \|\varphi\|_2^2 + \|\psi_x\|_2^2 + \|q\|_2^2,$$

as in chapter 4 (see page 72-75), one can easily show that B and F are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned}
B\big((\varphi,\psi,q),(\varphi,\psi,q)\big) &= K\int_0^1 (\varphi_x+\psi)^2dx + (\beta+\tau)\int_0^1 q^2dx + b\int_0^1 \psi_x^2dx \\
&\quad +\rho_2\int_0^1 \psi^2dx + \tilde{\mu}\int_0^1 \varphi^2dx + \rho_3(\beta+\tau)^2\int_0^1\left(\int_0^x q(y)dy\right)^2dx \\
&\geq \alpha_0\|(\varphi,\psi,q)\|_V^2,
\end{aligned}$$

for some $\alpha_0 > 0$. Thus B is coercive. Consequently, by Lax-Milgram Lemma, system (7.15) has a unique solution

$$\varphi \in H_0^1(0,1), \quad \psi \in H_\star^1(0,1) \quad \text{and} \quad q \in L_\star^2(0,1).$$

Moreover, if $(\varphi_1,\ q_1) \equiv (0,\ 0) \in H_0^1(0,1) \times L_\star^2(0,1)$, then (7.16) reduces to

$$\begin{aligned} K\int_0^1 (\varphi_x + \psi)\,\psi_1 dx + b\int_0^1 \psi_x \psi_{1x} dx + \rho_2 \int_0^1 \psi\psi_1 dx - \delta\,(\beta+\tau)\int_0^1 q\psi_1 dx \\ = \int_0^1 h_2 \psi_1 dx, \qquad \forall \psi_1 \in H_\star^1(0,1). \end{aligned}$$

Now, by following the same steps on page 105, we get

$$b\psi_{xx} = K\,(\varphi_x + \psi) + \rho_2 \psi - (\beta+\tau)\,\delta q - h_2 \in L^2(0,1),$$

which solves $(7.15)_2$. Consequently, by the elliptic regularity theory, it follows that

$$\psi \in H^2(0,1) \cap H_\star^1(0,1).$$

Furthermore, by following the same steps on page 107, we obtain

$$\psi_x(0,t) = \psi_x(1,t) = 0.$$

Thus, we get

$$\psi \in H_\star^2(0,1) \cap H_\star^1(0,1).$$

Similarly, we have

$$K\varphi_{xx} = \tilde{\mu}\varphi - h_1 + K\psi_x \ \in L^2(0,1)$$

and

$$q_x = (\beta+\tau)\,\rho_3 \int_0^x q(y)dy - \delta\psi_x - h_3 \ \in L^2(0,1),$$

and by the regularity of the elliptic problem, we can conclude that

$$\varphi \in H^2(0,1) \cap H_0^1(0,1) \quad \text{and} \quad q \in H_\star^1(0,1).$$

Furthermore, we deduce from (7.11)–(7.13) and (7.14) that

$$u \in H_0^1(0,1), \quad v \in H_\star^1(0,1), \quad \theta \in H_0^1(0,1), \text{ and } z, z_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

It is obvious from (7.11) and (7.14) that $z(x,0) = u(x)$.

Hence, there exists a unique $\Phi \in D(\mathcal{A})$ such that (7.9) is satisfied. Therefore, the operator $\mathcal{A}$ is maximal. With this, we conclude that $\mathcal{A}$ is a maximal monotone operator. On the other hand, it is obvious that operator $\mathcal{B}$ is Lipschitz continuous (see similar proof on page 107 in chapter 5). Consequently, $\mathcal{A}+\mathcal{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$. Hence, the result of Theorem 7.1 follows (see [46, 93]). ∎

7.3 Exponential Decay Result

In this section, we discuss the asymptotic behavior of the solution of problem (7.6). We define the energy functional

$$
\begin{aligned}
E(t) = & \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + b\psi_x^2 + K\left(\varphi_x + \psi\right)^2 + \tau q^2\right] dx \\
& + \frac{|\mu|\tau_0}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx.
\end{aligned} \tag{7.17}
$$

The following theorem is the main result of this section:

Theorem 7.2 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of* (7.6) *and assume that*

$$
\xi = \left(\tau - \frac{\rho_1}{K\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{K}\right) - \frac{\tau\rho_1\delta^2}{bK\rho_3} = 0. \tag{7.18}
$$

Then, for small $|\mu|$, the energy functional (7.17) *satisfies*

$$
E(t) \le c_0 e^{-c_1 t}, \qquad \forall t \ge 0, \tag{7.19}
$$

where c_0 and c_1 are two positive constants.

To establish the proof of Theorem 7.2, we need several lemmas.

Lemma 7.1 *Let $(\varphi, \psi, \theta, q)$ be the solution of* (7.6). *Then the energy functionals, defined by* (7.17), *satisfies*

$$
E'(t) = -\beta\int_0^1 q^2 dx + |\mu|\int_0^1 \varphi_t^2 \qquad \forall t \ge 0. \tag{7.20}
$$

Proof. A simple multiplication of $(7.6)_1$, $(7.6)_2$, $(7.6)_3$ and $(7.6)_4$, by $\varphi_t, \psi_t, \theta$ and q, respectively, and integration over $(0,1)$, using integration by parts and the boundary conditions, yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigg\{\int_0^1 \rho_1\varphi_t^2 dx + K\int_0^1 (\varphi_x+\psi)^2\,dx + \rho_2\int_0^1 \psi_t^2 dx + b\int_0^1 \psi_x^2 dx \\ + \rho_3\int_0^1 \theta^2 dx + \tau\int_0^1 q^2 dx\bigg\} \\ = -\beta\int_0^1 q^2 dx - \mu\int_0^1 \varphi_t z(x,1,t)dx.\end{aligned} \tag{7.21}$$

Now, multiplying $(7.6)_5$ by $|\mu| z$ and integrating over $(0,1)\times(0,1)$, bearing in mind $(7.6)_6$, we obtain

$$\frac{|\mu|\tau_0}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx = \frac{|\mu|}{2}\int_0^1 \varphi_t^2 dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1,t)dx. \tag{7.22}$$

Equation (7.20) follows by the combination of (7.21)-(7.22) and the use of Young's inequality. ∎

Remark 7.1 It is obvious from (7.20) that the energy E is not decreasing in general. Thus, system (7.6) is not necessarily dissipative.

Lemma 7.2 *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (7.6). Then the functional*

$$F_1(t) := -\int_0^1 \left[\rho_1\varphi_t\varphi + \rho_2\psi_t\psi\right]dx$$

satisfies, for all $\varepsilon_1 > 0$, the estimate

$$
\begin{aligned}
F_1'(t) \leq & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + c\left(1+\frac{\mu^2}{\varepsilon_1}\right) \int_0^1 (\varphi_x+\psi)^2\, dx \\
& + c\left(1+\frac{\mu^2}{\varepsilon_1}\right) \int_0^1 \psi_x^2 dx + c\int_0^1 \theta^2 dx + \varepsilon_1 \int_0^1 z^2(x,1,t) dx.
\end{aligned}
\tag{7.23}
$$

Proof. Direct computations, using the $(7.6)_1$ and $(7.6)_2$, give

$$
\begin{aligned}
F_1'(t) = & -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + b\int_0^1 \psi_x^2 dx + K\int_0^1 (\varphi_x+\psi)^2\, dx \\
& - \delta \int_0^1 \psi_x \theta dx + \mu \int_0^1 \varphi z(x,1,t) dx.
\end{aligned}
$$

By using Young's inequality, for $\varepsilon_1 > 0$ and the fact that

$$
\int_0^1 \varphi_x^2 dx \leq 2\int_0^1 (\varphi_x+\psi)^2\, dx + 2\int_0^1 \psi_x^2 dx, \tag{7.24}
$$

the conclusion of Lemma 7.2 follows. ∎

Lemma 7.3 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (7.6). Then the functional*

$$
F_2(t) := -\frac{\rho_2\rho_3}{\delta} \int_0^1 \theta \left(\int_0^x \psi_t(y,t) dy\right) dx
$$

satisfies, for all $\varepsilon_2 > 0$, the estimate

$$
\begin{aligned}
F_2'(t) \leq & -\frac{\rho_2}{2} \int_0^1 \psi_t^2 dx + 2\varepsilon_2 \int_0^1 \psi_x^2 dx + \varepsilon_2 \int_0^1 (\varphi_x+\psi)^2\, dx \\
& + c\left(1+\frac{1}{\varepsilon_2}\right) \int_0^1 \theta^2 dx + c\int_0^1 q^2 dx.
\end{aligned}
\tag{7.25}
$$

Proof. Taking the derivative of F_2, using $(7.6)_2$, $(7.6)_3$ and recalling that ψ

stands for $\overline{\psi}$, it easily follows that

$$F_2'(t) = -\rho_2 \int_0^1 \psi_t^2 dx - \frac{\rho_2}{\delta} \int_0^1 q\psi_t dx - \frac{b\rho_3}{\delta} \int_0^1 \theta\psi_x dx + \frac{K\rho_3}{\delta} \int_0^1 \theta\varphi dx$$
$$+ \rho_3 \int_0^1 \theta^2 dx + \frac{K\rho_3}{\delta} \int_0^1 \theta \left(\int_0^x \psi(y,t)dy \right) dx.$$

By exploiting Cauchy-Schwarz, Poincaré's and Young's inequalities, for $\varepsilon_2 > 0$, we obtain

$$F_2'(t) \leq -\frac{\rho_2}{2} \int_0^1 \psi_t^2 dx + \varepsilon_2 \int_0^1 \psi_x^2 dx + \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx$$
$$+ c\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 \theta^2 dx + c \int_0^1 q^2 dx.$$

Estimate (7.25) is established thanks to (7.24), ▮

Lemma 7.4 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (7.6). Then the functional*

$$F_3(t) := \rho_2 \int_0^1 \psi\psi_t - \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi(y,t)dy \right) dx$$

satisfies, for all $\varepsilon_3 > 0$, the estimate

$$F_3'(t) \leq -\frac{b}{2} \int_0^1 \psi_x^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_3}\right) \int_0^1 \psi_t^2 dx$$
$$+ c \int_0^1 \theta^2 dx + \frac{\mu^2}{b} \int_0^1 z^2(x,1,t) dx. \tag{7.26}$$

Proof. A simple differentiation of F_3, using $(7.6)_1$ and $(7.6)_2$, gives

$$F_3'(t) = -b\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx + \delta \int_0^1 \psi_x \theta dx - \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi_t(y,t)dy \right) dx$$
$$+ \mu \int_0^1 z(x,1,t) \left(\int_0^x \psi(y,t)dy \right) dx.$$

By recalling Cauchy-Schwarz, Poincaré's and Young's inequalities, for $\varepsilon_3 > 0$, estimate (7.26) is established. ∎

Lemma 7.5 *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of* (7.6)*. Then the functional*

$$F_4(t) := \tau\rho_3 \int_0^1 \theta \left(\int_0^x q(y,t)dy \right) dx$$

satisfies, for all $\varepsilon_4 > 0$, *the estimate*

$$F_4'(t) \leq -\frac{\rho_3}{2} \int_0^1 \theta^2 dx + \varepsilon_4 \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_4}\right) \int_0^1 q^2 dx. \tag{7.27}$$

Proof. Differentiation of F_4, then using $(7.6)_3, (7.6)_4$, integration by parts and using the fact that $\int_0^1 q(x,t)dx = 0$ (it is to be noted that we are working with $\overline{q}$) give

$$F_4'(t) = -\rho_3 \int_0^1 \theta^2 dx + \tau \int_0^1 q^2 dx + \tau\delta \int_0^1 q\psi_t dx - \beta\rho_3 \int_0^1 \theta \left(\int_0^x q(y,t)dy \right) dx.$$

Consequently, (7.27) follows by using Cauchy-Schwarz and Young's inequalities for any $\varepsilon_4 > 0$. ∎

Next, we define another functional bearing in mind the definition of ξ as given in (7.18).

Lemma 7.6 *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (7.6) and assume that $\xi = 0$. Then the functional*

$$
\begin{aligned}
F_5(t) := \frac{\tau\rho_2}{K}\int_0^1 \psi_t\left(\varphi_x+\psi\right)dx + \frac{b\tau\rho_1}{K^2}\int_0^1 \varphi_t\psi_x dx - \frac{b\tau\rho_3}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 \theta\varphi_t dx \\
+ \frac{b\tau}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 q\left(\varphi_x+\psi\right)dx
\end{aligned}
$$

satisfies, for all $\varepsilon_5 > 0$, the estimate

$$
\begin{aligned}
F_5'(t) \leq & -\frac{\tau}{2}\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \varepsilon_5\int_0^1 z^2(x,1,t)dx \\
& + c\left(\int_0^1 \psi_t^2 dx + \int_0^1 q^2 dx + \frac{\mu^2}{\varepsilon_5}\int_0^1 \psi_x^2 dx + \frac{\mu^2}{\varepsilon_5}\int_0^1 \theta^2 dx\right).
\end{aligned} \tag{7.28}
$$

Proof. A simple differentiation of F_5 gives

$$
\begin{aligned}
F_5'(t) = & \frac{\tau\rho_2}{k}\int_0^1 \psi_{tt}\left(\varphi_x+\psi\right)dx - \frac{b\tau\rho_3}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 \theta\varphi_{tt}dx \\
& + \frac{\tau\rho_2}{K}\int_0^1 \psi_t^2 dx + \frac{b\tau\rho_1}{K^2}\int_0^1 \varphi_{tt}\psi_x dx + \frac{b\tau}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 q_t\left(\varphi_x+\psi\right)dx \\
& + \frac{b\tau}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 q\varphi_{xt}dx + \frac{b\tau}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 q\psi_t dx \\
& - \frac{b\tau\rho_3}{\delta K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 \theta_t\varphi_t dx - \frac{b\tau}{K}\left(\frac{\rho_2}{b}-\frac{\rho_1}{K}\right)\int_0^1 \varphi_t\psi_{xt}dx.
\end{aligned} \tag{7.29}
$$

Now, we work out the terms in the right-hand side of (7.29), using integration by parts and the equations in (7.6).

$$\rho_2 \int_0^1 \psi_{tt} (\varphi_x + \psi)\, dx = -K \int_0^1 (\varphi_x + \psi)^2\, dx + b \int_0^1 \psi_{xx} (\varphi_x + \psi)\, dx \\ - \delta \int_0^1 \theta_x (\varphi_x + \psi)\, dx, \tag{7.30}$$

$$- \int_0^1 \theta \varphi_{tt} dx = \frac{K}{\rho_1} \int_0^1 \theta_x (\varphi_x + \psi)\, dx + \frac{\mu}{\rho_1} \int_0^1 \theta z(x, 1, t) dx, \tag{7.31}$$

$$\rho_1 \int_0^1 \varphi_{tt} \psi_x dx = -K \int_0^1 \psi_{xx} (\varphi_x + \psi)\, dx - \mu \int_0^1 \psi_x z(x, 1, t) dx, \tag{7.32}$$

$$\tau \int_0^1 q_t (\varphi_x + \psi)\, dx = -\beta \int_0^1 q (\varphi_x + \psi)\, dx - \int_0^1 \theta_x (\varphi_x + \psi)\, dx \tag{7.33}$$

and

$$-\rho_3 \int_0^1 \theta_t \varphi_t dx = - \int_0^1 q \varphi_{xt} dx + \delta \int_0^1 \varphi_t \psi_{xt} dx. \tag{7.34}$$

The subsitution of (7.30)–(7.34) into (7.29), gives

$$\begin{aligned} F_5'(t) = &-\tau \int_0^1 (\varphi_x + \psi)^2\, dx + \frac{\tau \rho_2}{K} \int_0^1 \psi_t^2 dx + \frac{b\tau}{\delta K} \left(\frac{\rho_2}{b} - \frac{\rho_1}{K} \right) \int_0^1 q \psi_t dx \\ &- \frac{b\beta}{\delta K} \left(\frac{\rho_2}{b} - \frac{\rho_1}{K} \right) \int_0^1 q (\varphi_x + \psi)\, dx - \frac{\mu b \tau}{K^2} \int_0^1 \psi_x z(x, 1, t) dx \\ &+ \frac{\mu b \tau \rho_3}{K \delta \rho_1} \left(\frac{\rho_2}{b} - \frac{\rho_1}{K} \right) \int_0^1 \theta z(x, 1, t) dx \\ &+ \frac{b \rho_3}{\delta \rho_1} \left[\left(\tau - \frac{\rho_1}{K \rho_3} \right) \left(\frac{\rho_2}{b} - \frac{\rho_1}{K} \right) - \frac{\tau \rho_1 \delta^2}{b K \rho_3} \right] \int_0^1 \theta_x (\varphi_x + \psi)\, dx. \end{aligned} \tag{7.35}$$

By recalling that $\xi = 0$ and using Young's inequality, for $\varepsilon_5 > 0$, (7.28) follows. ▌

Lemma 7.7 *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of* (7.6). *Then the functional*

$$F_6(t) := \tau_0 \int_0^1 \int_0^1 e^{-\tau_0 \rho} z^2(x, \rho, t) d\rho dx$$

satisfies, for some positive constant m_0, the estimate

$$F_6' \le -m_0 \left(\int_0^1 z^2(x,1,t)dx + \tau_0 \int_0^1 \int_0^1 z^2(x,\rho,t)d\rho dx \right) + \int_0^1 \varphi_t^2 dx. \qquad (7.36)$$

Proof. See the proof of Lemma 3.4 on page 45. ∎

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Lemma 7.8 *For N sufficiently large, the functional defined by*

$$\mathcal{L}(t) := NE(t) + F_1(t) + N_1F_2(t) + N_2F_3(t) + N_3F_4(t) + N_4F_5 + \frac{\rho_1}{2}F_6, \qquad (7.37)$$

where N_i are positive real numbers to be chosen appropriately later, satisfies

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t), \ \forall t \ge 0, \qquad (7.38)$$

for two positive constants α_1 and α_2.

Proof. The lemma is established by following the same steps enumerated in the proof of Lemma 3.6 (see page 50). ∎

Proof of Theorem 7.2.

We differentiate (7.37) and recall (7.20), (7.23), (7.25)–(7.28) and (7.36) to obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\beta N - cN_1 - cN_3\left(1+\frac{1}{\varepsilon_4}\right) - cN_4\right]\int_0^1 q^2 dx \\
& - \left[\frac{\rho_1}{2} - \varepsilon_3 N_2 - |\mu| N\right]\int_0^1 \varphi_t^2 dx - \frac{\rho_1 m_0 \tau_0}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx \\
& - \left[\frac{\rho_2}{2}N_1 + \rho_2 - \varepsilon_4 N_3 - cN_2\left(1+\frac{1}{\varepsilon_3}\right) - cN_4\right]\int_0^1 \psi_t^2 dx \\
& - \left[\frac{b}{2}N_2 - 2\varepsilon_2 N_1 - c\left(1+\frac{\mu^2}{\varepsilon_1}\right) - \frac{c\mu^2}{\varepsilon_5}N_4\right]\int_0^1 \psi_x^2 dx \\
& - \left[\frac{\rho_3}{2}N_3 - c - cN_1\left(1+\frac{1}{\varepsilon_2}\right) - cN_2 - \frac{c\mu^2}{\varepsilon_5}N_4\right]\int_0^1 \theta^2 dx \\
& - \left[\frac{\tau}{2}N_4 - \varepsilon_2 N_1 - c\left(1+\frac{\mu^2}{\varepsilon_1}\right)\right]\int_0^1 (\varphi_x+\psi)^2 dx \\
& - \left[\frac{\rho_1 m_0}{2} - \varepsilon_1 - \varepsilon_5 N_4 - \frac{\mu^2}{b}N_2\right]\int_0^1 z^2(x,1,t)dx.
\end{aligned}
$$

Now, we need to choose carefully our constants. We set

$$
\varepsilon_3 = \frac{\rho_1}{4N_2},\ \varepsilon_4 = \frac{\rho_2}{N_3},\ \varepsilon_2 = \frac{\tau N_4}{4N_1},\ \varepsilon_1 = \frac{m_0\rho_1}{4} \ \text{ and } \ \varepsilon_5 = \frac{\rho_1 m_0}{8N_4}.
$$

This choice yields

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\beta N - cN_1 - cN_3\left(1+N_3\right) - cN_4\right]\int_0^1 q^2 dx \\
& - \left[\frac{\rho_2}{2}N_1 - cN_2\left(1+N_2\right) - cN_4\right]\int_0^1 \psi_t^2 dx \\
& - \left[\frac{b}{2}N_2 - c\left(1+\mu^2\right) - cN_4\left(1+\mu^2 N_4\right)\right]\int_0^1 \psi_x^2 dx \\
& - \left[\frac{\rho_3}{2}N_3 - c - cN_1\left(1+\frac{N_1}{N_4}\right) - cN_2 - c\mu^2 N_4^2\right]\int_0^1 \theta^2 dx \\
& - \left[\frac{\tau}{4}N_4 - c\left(1+\mu^2\right)\right]\int_0^1 (\varphi_x+\psi)^2 dx - \left[\frac{\rho_1}{4} - |\mu| N\right]\int_0^1 \varphi_t^2 dx
\end{aligned}
$$

$$- \left[\frac{\rho_1 m_0}{8} - \frac{\mu^2}{b} N_2\right] \int_0^1 z^2(x,1,t)dx - \frac{\rho_1 m_0 \tau_0}{2} \int_0^1 \int_0^1 z^2(x,\rho,t)d\rho dx.$$

At this point, we choose N_4 large enough so that

$$k_1 = \frac{\tau}{4} N_4 - c > 0,$$

then we choose N_2 large enough so that

$$k_2 = \frac{b}{2} N_2 - c - cN_4 > 0,$$

then we choose N_1 large enough so that

$$k_3 = \frac{\rho_2}{2} N_1 - cN_2 (1 + N_2) - cN_4 > 0,$$

then we choose N_3 large enough so that

$$k_4 = \frac{\rho_3}{2} N_3 - c - cN_1 \left(1 + \frac{N_1}{N_4}\right) - cN_2 > 0,$$

then we choose N large enough (even larger) so that (7.38) remains valid and, furthermore,

$$k_5 = \beta N - cN_1 - cN_3 (1 + N_3) - cN_4 > 0.$$

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - k_5 \int_0^1 q^2 dx - k_3 \int_0^1 \psi_t^2 dx - \left[k_2 - c\mu^2\right] \int_0^1 \psi_x^2 dx \\
& - \left[k_1 - c\mu^2\right] \int_0^1 (\varphi_x + \psi)^2 \, dx - \frac{\rho_1 m_0 \tau_0}{2} \int_0^1 \int_0^1 z^2(x,\rho,t) d\rho dx \\
& - \left[\frac{\rho_1}{4} - c|\mu|\right] \int_0^1 \varphi_t^2 dx - \left[k_4 - c\mu^2\right] \int_0^1 \theta^2 dx \\
& - \left[\frac{\rho_1 m_0}{8} - c\mu^2\right] \int_0^1 z^2(x,1,t) dx.
\end{aligned}
$$

Finally, by taking $|\mu|$ so small that

$$
k_1 - c\mu^2 > 0, \;\; k_2 - c\mu^2 > 0, \;\; k_4 - c\mu^2 > 0, \;\; \frac{\rho_1}{4} - c|\mu| > 0, \;\; \frac{\rho_1 m_0}{8} - c\mu^2 > 0,
$$

and using (7.17), we get

$$
\mathcal{L}'(t) \leq -c_2 E(t), \qquad \forall t > 0, \tag{7.39}
$$

where c_2 is a positive constant.

A combination of (7.38) and (7.39) gives

$$
\mathcal{L}'(t) \leq -c_1 \mathcal{L}(t), \qquad \forall t > 0, \tag{7.40}
$$

where $c_1 = \dfrac{c_2}{\alpha_2}$. A simple integration of (7.39) over $(0,t)$ yields

$$
\mathcal{L}(t) \leq \mathcal{L}(0) e^{-c_1 t}, \qquad \forall t > 0. \tag{7.41}
$$

Thus, using (7.38) and (7.41), the conclusion of Theorem 7.2 follows.

Remark 7.2 As in Pignotti [95], one can consider the auxiliary problem

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + |\mu|\varphi_t + \mu\varphi_t(x, t-\tau_0) = 0, & x \in (0,1),\ t > 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & x \in (0,1),\ t > 0 \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & x \in (0,1),\ t > 0 \\ \tau q_t + \beta q + \theta_x = 0, & x \in (0,1),\ t > 0 \\ \varphi_t(x, -t) = f_0(x,t), & x \in (0,1),\ t \in (0,\tau) \\ \varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, & \forall t \geq 0 \end{cases} \tag{7.42}$$

and establish an exponential decay result by either repeating the above calculations (noting that the energy for (7.42) is dissipative) or by applying the semigroup theory as in Santos *et al.* [108]. Then our problem (7.1) can be regarded as a bounded perturbation of the auxiliary problem (7.42). Consequently, the stability result will also hold for our problem, for μ small enough (See Pazy [93], chapter III).

7.4 Polynomial Decay Result (for $\mu = 0,\ \ \xi \neq 0$)

In this section, we establish the polynomial decay result for system (7.1) and (7.2) when $\mu = 0$, using the multiplier method instead of the semigroup method used in [108]. Therefore, we consider the following problem:

$$
\begin{cases}
\rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0, & x \in (0,1),\ t > 0, \\
\rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \delta\theta_x = 0, & x \in (0,1),\ t > 0, \\
\rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & x \in (0,1),\ t > 0, \\
\tau q_t + \beta q + \theta_x = 0, & x \in (0,1),\ t > 0, \\
\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0,\ \forall t \geq 0.
\end{cases} \tag{7.43}
$$

The first-order energy functional for the solution of problem (4.1), is given by

$$
E_1(t) = \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + b\psi_x^2 + K\left(\varphi_x + \psi\right)^2 + \tau q^2\right] dx. \tag{7.44}
$$

Furthemore, for any strong solution, we define the second-order energy functional

$$
E_2(t) = \frac{1}{2}\int_0^1 \left[\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \rho_3\theta_t^2 + b\psi_{tx}^2 + K\left(\varphi_{tx} + \psi_t\right)^2 + \tau q_t^2\right] dx. \tag{7.45}
$$

It is very easy to verify that the energy functionals defined by (7.44) and (7.45), satisfy,

$$
E_1'(t) = -\beta\int_0^1 q^2 dx, \quad \forall t \geq 0 \tag{7.46}
$$

and

$$
E_2'(t) = -\beta\int_0^1 q_t^2 dx, \quad \forall t \geq 0. \tag{7.47}
$$

The following theorem is the main result of this section:

Theorem 7.3 *Let $(\varphi, \psi, \theta, q)$ be the strong solution of (4.1) and assume that*

$$\xi = \left(\tau - \frac{\rho_1}{K\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{K}\right) - \frac{\tau\rho_1\delta^2}{bK\rho_3} \neq 0. \tag{7.48}$$

Then there exists a positive constant λ_0 such that the energy functional (7.44) satisfies,

$$E_1(t) \leq \frac{\lambda_0}{t}, \quad \forall t > 0. \tag{7.49}$$

As in (7.37), we define a Lyapunov functional $\mathcal{L}$ as follows:

$$\mathscr{L}(t) := N(E_1(t)+E_2(t))+F_1(t)+N_1F_2(t)+N_2F_3(t)+N_3F_4(t)+N_4F_5(t), \tag{7.50}$$

where $F_i,\ i = 1-5$, remain as defined in Lemma 7.2–Lemma 7.6 but with derivatives given as

$$F_1'(t) \leq -\rho_1 \int_0^1 \varphi_t^2 dx - \rho_2 \int_0^1 \psi_t^2 dx + c\int_0^1 (\varphi_x + \psi)^2 dx + c\int_0^1 \psi_x^2 dx + c\int_0^1 \theta^2 dx, \tag{7.51}$$

$$\begin{aligned} F_2'(t) \leq -\frac{\rho_2}{2}\int_0^1 \psi_t^2 dx + 2\varepsilon_2 \int_0^1 \psi_x^2 dx + \varepsilon_2 \int_0^1 (\varphi_x + \psi)^2 dx \\ + c\left(1 + \frac{1}{\varepsilon_2}\right)\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx, \end{aligned} \tag{7.52}$$

$$F_3'(t) \leq -\frac{b}{2}\int_0^1 \psi_x^2 dx + \varepsilon_3 \int_0^1 \varphi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_3}\right)\int_0^1 \psi_t^2 dx + c\int_0^1 \theta^2 dx, \tag{7.53}$$

$$F_4'(t) \leq -\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_4 \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_4}\right)\int_0^1 q^2 dx \tag{7.54}$$

and

$$F_5'(t) \leq -\frac{\tau}{2}\int_0^1 (\varphi_x+\psi)^2\,dx + c\left(\int_0^1 \psi_t^2 dx + \int_0^1 q^2 dx + \int_0^1 q_t^2 dx\right). \qquad (7.55)$$

Estimates (7.51)–(7.55) can easily be obtained by following the steps in the proof of Lemma 7.2–Lemma 7.6. For example, (7.55) easily follows from (7.35) by using Young's inequality and the fact that

$$\int_0^1 \theta_x^2 dx \leq c\left(\int_0^1 q^2 dx + \int_0^1 q_t^2 dx\right),$$

thanks to $(7.43)_4$.

Remark 7.3 It is important to note that $\mathscr{L} \not\sim E_1$. In other words, (7.38) no longer holds.

Proof of Theorem 7.3. To finalize the proof of Theorem 7.3, we differentiate the Lyapunov functional $\mathscr{L}(t)$ defined in (7.50) and use (7.46)–(7.47), (7.51)–(7.55), to obtain

$$\begin{aligned}
\mathscr{L}'(t) \leq \; & - \; \left[\beta N - cN_1 - cN_3\left(1+\frac{1}{\varepsilon_4}\right) - cN_4\right]\int_0^1 q^2 dx - \left[\beta N - cN_4\right]\int_0^1 q_t^2 dx \\
& - \; \left[\frac{\rho_2}{2}N_1 + \rho_2 - \varepsilon_4 N_3 - cN_2\left(1+\frac{1}{\varepsilon_3}\right) - cN_4\right]\int_0^1 \psi_t^2 dx \\
& - \; \left[\rho_1 - \varepsilon_3 N_2\right]\int_0^1 \varphi_t^2 dx - \left[\frac{b}{2}N_2 - 2\varepsilon_2 N_1 - c\right]\int_0^1 \psi_x^2 dx \\
& - \; \left[\frac{\rho_3}{2}N_3 - c - cN_1\left(1+\frac{1}{\varepsilon_2}\right) - cN_2\right]\int_0^1 \theta^2 dx \\
& - \; \left[\frac{\tau}{2}N_4 - \varepsilon_2 N_1 - c\right]\int_0^1 (\varphi_x+\psi)^2\,dx.
\end{aligned}$$

As in the proof of Theorem 7.1, we set

$$\varepsilon_2 = \frac{\tau N_4}{4N_1}, \;\; \varepsilon_3 = \frac{\rho_1}{4N_2} \;\; \text{and} \;\; \varepsilon_4 = \frac{\rho_2}{N_3}.$$

This choice yields

$$\begin{aligned}
\mathscr{L}'(t) \leq \;\; & - \;\; \left[\beta N - cN_1 - cN_3\left(1+N_3\right) - cN_4\right]\int_0^1 q^2dx - \left[\beta N - cN_4\right]\int_0^1 q_t^2dx \\
& - \;\; \left[\frac{\rho_2}{2}N_1 - cN_2\left(1+N_2\right) - cN_4\right]\int_0^1 \psi_t^2dx - \frac{3\rho_1}{4}\int_0^1 \varphi_t^2dx \\
& - \;\; \left[\frac{b}{2}N_2 - \frac{\tau N_4}{2} - c\right]\int_0^1 \psi_x^2dx - \left[\frac{\tau}{4}N_4 - c\right]\int_0^1 \left(\varphi_x+\psi\right)^2dx \\
& - \;\; \left[\frac{\rho_3}{2}N_3 - c - cN_1\left(1+\frac{N_1}{N_4}\right) - cN_2\right]\int_0^1 \theta^2dx.
\end{aligned}$$

In addition to the same choice of N and N_i $(i = 1, ..., 5)$ as in the proof of Theorem 7.1, we further choose N large enough so that

$$\beta N - cN_4 > 0.$$

Thus, we get

$$\mathscr{L}'(t) \leq -\lambda_1 E_1(t) \quad \forall t > 0, \tag{7.56}$$

where λ_1 is a positive constant. A simple integration of (7.56) over $(0,t)$, recalling that E_1 is nonincreasing, yields

$$tE_1(t) \leq \int_0^t E_1(s)ds \leq \frac{1}{\lambda_1}\left(\mathscr{L}(0) - \mathscr{L}(t)\right) \leq \frac{\mathscr{L}(0)}{\lambda_1}.$$

Finally, for $\lambda_0 = \frac{\mathscr{L}(0)}{\lambda_1} = \frac{E_1(0)+E_2(0)}{\lambda_1}$, we have

$$E_1(t) \leq \frac{\lambda_0}{t} \quad \forall t > 0,$$

which completes the proof.

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